International Journal of Theoretical Physics, Vol. 25, No. 6, 1986

Cosmological Limits on Computation

Frank J. Tipler[1]

Received October 5, 1985

A true universal Turing machine can be constructed only if it is possible to actually process and store an infinite number of bits between now and the end of the universe. Conditions on the universe are derived that must hold if such processing and storage is to be possible. In particular, it is shown that it is possible only if the universe is closed and only if its future c-boundary consists of a single point.

1. INTRODUCTION

In the last decade, there has been enormous progress in our understanding of the fundamental physical limitations on computation (Landauer, 1985; Bennett and Landauer, 1985). However, this work has been entirely concerned with limitations arising from local physical laws, most notably the second law of thermodynamics and the uncertainty relations applied in a noncosmological context. As Bennett and Landauer (1985) point out at the end of their recent nontechnical review of this work, the ultimate physical limitations on computation will likely arise from limitations on the bit size of computer memories and the speed with which different parts of the computer can communicate with each other. The ultimate limit to physical size is the size of the entire universe, and the ultimate speed is that imposed by relativity. Thus, the ultimate physical limitations are those imposed by cosmology and relativity.

In this paper I shall investigate the limitations imposed on computation by general relativity in the cosmological context. In Section 2, I shall review some of the basic features of relativistic cosmology. The global causal structure of the actual universe is expected to be quite different from that

[1]Departments of Mathematics and Physics, Tulane University, New Orleans, Louisiana 70118.

of Minkowski space, and this will drastically effect the speed-of-light limitation to computation. Furthermore, I shall point out that the most abundant energy source for the recording of computer output [no energy is required for processing if the processing is sufficiently slow (Landauer, 1985)] and the manufacture of computer elements is cosmological gravitational anisotropy. In Section 3, I review how the material content of the universe is expected to evolve in the far future. This is necessary in order to determine which elementary particles will be available for the construction of computer elements. In Section 4, I state necessary conditions for an infinite number of bits of information to be processed between now and the end of the universe. I show that the known physical laws will permit these necessary conditions to be satisfied, provided the universe is closed, the final singularity is a single point in the *c*-boundary topology, and the elementary particle spectrum has a certain form.

The material in this paper is covered from a slightly different point of view in the tenth chapter of my recent book with John D. Barrow of the University of Sussex (Barrow and Tipler, 1986). I shall assume in the following sections that the Einstein equations without cosmological constant hold; that is, $R_{ab}-\frac{1}{2}g_{ab}R=8\pi T_{ab}$. The signature of spacetime is $(-+++)$; the notation and other conventions will be that of Hawking and Ellis (1973).

2. COSMOLOGICAL MODELS AND THE GLOBAL STRUCTURE OF SPACETIME

For simplicity I restrict attention to cosmologies that possess a Cauchy hypersurface. A *Cauchy hypersurface* is defined to be a hypersurface that every timelike curve intersects exactly once (Hawking and Ellis, 1973). Spacetimes that possess a Cauchy hypersurface can be shown to be globally hyperbolic (Hawking and Ellis, 1973), and all globally hyperbolic spacetimes possess Cauchy hypersurfaces, so "globally hyperbolic" and "possessing a Cauchy hypersurface" are equivalent. A spacetime is said to be *globally hyperbolic* if (1) all sets of the form $J^+(p)\cap J^-(q)$ are compact for all events p and q in the spacetime, and further (2) strong causality holds. The set $J^+(p)$ is the *causal future* of the event p, and is the set of all events that can be reached from p by a future-directed timelike or null curve. Thus $J^+(p)$ is the set of all events that can be influenced by p. The set $J^-(q)$ is the *causal past* of the event q, and is defined analogously to $J^+(p)$. The set $J^-(p)$ is the set of all events that can influence the event p. [The sets $I^+(p)$ and $I^-(p)$ are the *chronological future* and *chronological past* of p, respectively. $I^+(p)$ is the set of all events that can be reached from p by a future-directed timelike curve. $I^+(p)$ is the set of all events that can be reached from p by a physical observer.] *Strong causality* is said to hold if every neighborhood of every event has a neighborhood that no timelike or null curve intersects more than once. The importance of the requirement of global hyperbolicity is that only globally hyperbolic spacetimes are uniquely determined by initial data given on an initial spacelike hypersurface (the Cauchy hypersurface) (Hawking and Ellis, 1973).

Geroch (1970) has shown (see also Hawking and Ellis, 1973) that globally hyperbolic spacetimes have topology $S \times R^1$, where S is the topology of the spacelike Cauchy hypersurface (all Cauchy hypersurfaces in a spacetime have the same topology), and R^1 is the time dimension. Globally hyperbolic cosmologies can be distinguished by the topology of their Cauchy hypersurfaces. The *closed universes* are those whose spacelike Cauchy hypersurfaces are compact (and without boundary), and *open universes* are those whose spacelike hypersurfaces are noncompact.

Cosmologies can also be classified in terms of their long-term dynamical evolution. Cosmologies whose size or radius of curvature (scale factor) grow without limit are called *ever-expanding universes*, while cosmologies that reach a maximum size and recollapse are called *recollapsing universes.*

In the popular literature, closed universes are pictured as always recollapsing, while open universes are considered always to be ever-expanding. This is due to the fact that the most familiar cosmological models—the Friedman models—do have these properties. Firedman cosmologies are, however, very special: by definition, their Cauchy hypersurfaces are homogeneous and isotropic. Their Cauchy hypersurfaces are generally considered to have two possible topologies, S^3 and R^3, the former being compact and the latter being noncompact. If the cosmological constant is zero and the stress energy tensor T_{ab} satisfies

$$(T_{ab} - \tfrac{1}{2} g_{ab} T) V^a V^b \geq 0 \tag{1}$$

for all timelike vectors V^a, if the inequality (1) is strict at least one point on every Cauchy hypersurface, with V^a being the normal to the hypersurface, and if the principle pressures of T_{ab} are all positive, then all open Friedman universes do indeed ever-expand, and all closed Friedman universes do indeed recollapse. Condition (1), which is called the *strong energy condition* (Hawking and Ellis, 1973), is satisfied by all physically realistic matter fields. The strict inequality just means that the matter density is nonzero at least one point in the entire universe, again a reasonable requirement.

But it is not known whether this connection between the topology of the Cauchy hypersurfaces and the long-term dynamical evolution holds for universes that are nonhomogeneous or anisotropic. This is important, because, as we shall see, in its actual evolution, the universe is likely to become extremely anisotropic.

A few general results are known about the long-term evolution of the universe. For example, it is possible to prove that a closed universe cannot return arbitrarily closely to a previous state:

Theorem 1. If a spacetime containing compact Cauchy hypersurfaces is uniquely developed from initial data on any of its Cauchy hypersurfaces, and if the strong energy condition holds with the inequality being strict at at least one event on each nonspacelike geodesic, then the spacetime cannot be time periodic. Furthermore, if the matter fields satisfy the standard conditions (a), (b), and (c) given in Hawking and Ellis (1973, pp. 254–255) for unique time development, then for any neighbourhood U of any Cauchy hypersurface S_1, a number e exists such that

$$\|(h, x, \chi, \chi') - (h_1, x_1, \chi_1, \chi_1')\|_{a+5} > e$$

for the initial data on any Cauchy hypersurface S with $U \cap S$ empty, where $\|\cdot\cdot\cdot\|_{a+5}$ is the Sobolev norm on initial data space, and (h, x, χ, χ') are the initial data on S.

The proof of the theorem is given in Tipler (1980), and a discussion of the meaning of the various conditions is given in (Tipler, 1979a). In brief, the Sovolev norm is a natural way of defining "nearness" for initial data sets for continuous fields. It amounts to comparing the differences of the field initial values averaged over the entire Cauchy hypersurface. If the initial data on two surfaces are truly different, this average should be nonzero, and Theorem 1 tells us that it is.

Theorem 1 shows us that general relativistic cosmology is quite different from Newtonian cosmology. In Newtonian mechanics, as is well known [see Tipler (1980) for a reveiw], a system constrained within a finite region of phase space is required to return arbitrarily closely to almost all previous states. This result, the Poincaré recurrence theorem, makes a closed universe cosmology based on Newtonian mechanics cyclic, or nonevolutionary, in the long run. It will be important for unlimited computation in a closed universe that general relativity forbids recurrence. With recurrence, a potentially infinite state machine such as a universal Turing machine would be impossible.

Another theorem, due collectively to Brill and Flaherty (1976), Marsden and Tipler (1980), and Gerhardt (1983) [see Barrow and Tipler (1985) for a review of these results] is that in a globally hyperbolic universe that is not everywhere flat and satisfies the strong energy condition, there will exist a unique globally defined time coordinate. A time coordinate in relativity is defined by a "slicing" of four-dimensional space–time by a sequence of three-dimensional spacelike hypersurfaces. This sequence is called a *foliation* of space–time, and each hypersurface is called a *leaf* of the foliation.

For a simple example of the concept of "foliation" consider the surface of an ordinary cylinder. The surface of an ordinary cylinder is two-dimensional, and it can be foliated by a sequence of circles perpendicular to the axis of the cylinder. The cylinder is then just all of these circles stacked on top of one another. Each circle is a leaf of the foliation, and the foliation is all of the circles together.

Any physically realistic cosmology can be foliated uniquely by Cauchy hypersurfaces of constant mean extrinsic curvature, and it is this foliation that defines the unique global time. The *extrinsic curvature* of a spacelike hypersurface is its relative rate of expansion in time. This relative rate of expansion is measured by the *Hubble parameter* $H \equiv (1/R)\, dR/dt$ used extensively in describing the Friedman universe. However, in a general cosmology it is possible for the universe to expand faster in some directions than others, so the Hubble parameter must be generalized into a tensor in order to express this direction dependence properly. This tensor is the extrinsic curvature. The *mean extrinsic curvature* is a scalar like the Hubble parameter, and it is an average of the extrinsic curvatures in the three spatial directions [more exactly, it is the contraction of the extrinsic curvature, which is a rank-two tensor; see Hawking and Ellis (1973) or Marsden and Tipler (1980) for a precise definition]. A *constant mean curvature hypersurface* is a spacelike hypersurface on which the mean extrinsic hypersurface is the same at every point. The hypersurfaces of homogeneity and isotropy in the Friedman universe are constant mean curvature hypersurfaces in which the mean curvature is $3H$. Since the universe is in fact closely isotropic and homogeneous, the constant mean hypersurface defining the global instant "now" over the entire universe essentially coincides with the spacelike hypersurface in which the 3 K background radiation is constant. The Earth is currently moving at about 350 km/sec with respect to this globally defined rest frame of the universe.

The most important physical factor in the dynamical evolution is the scale factor $R(t)$, which is a rough measure of the size of the universe. For spacetimes that are approximately homogeneous and isotropic over most of their history, the time behavior of $R(t)$ is governed by a generalized Friedman equation (Ellis, 1973; Barrow and Turner, 1970) along with an evolution equation for the shear tensor σ_{ab} (which is just the trace-free part of the extrinsic curvature) of the timelike geodesic congruence normal to the Cauchy hypersurfaces (the shear σ is defined by $\sigma^2 = \sigma_{ab}\sigma^{ab} \geq 0$):

$$\dot{R}^2/R^2 = (8\pi G/3)(\rho_m + \rho_r) + \sigma^2 - k/R^2 \tag{2}$$

$$\sigma_a^b = R^{-3}\left\{\Sigma^{ab} + \int [{}^{(3)}R_a^b - \tfrac{1}{3}\delta_a^b\, {}^{(3)}R]R^{-3}\, dt\right\} \tag{3}$$

where Σ_{ab} is time-independent ($\Sigma_{ab}\Sigma^{ab} \equiv \Sigma^2$) and the quantity in (3) under the integral is the anisotropic part of the spatial three-curvature (the three-curvature of the Cauchy hypersurface is written ${}^{(3)}R$); $k=+1$ if the universe has spatial topology S^3, and $k=-1$ or 0 if it has the topology R^3 ($k=0$ if and only if the Cauchy hypersurfaces are flat). Cosmologists will often reserve the term "open universe" for the $k=-1$ open universes, and call the $k=0$ open universes "flat universes." The term σ^2 measures the energy in the form of anisotropic gravitational shear, or, roughly speaking, the energy in the form of very long-wavelength gravitational waves.

The shear evolution equation (3) shows that, in general, σ has two sources, a kinematic Newtonian component Σ^2/R^{-6} associated with the isotropic part of the curvature, and a non-Newtonian part associated with the spatial curvature anisotropy. In the most general ever-expanding anisotropic cosmological models it is this anisotropic curvature term that tends to dominate the dynamics at late times. In these open models, it typically contributes a shear evolution $\sigma^2 \propto R^{-2}$ as its dominant term at late times. In the recollapsing closed models, the $\Sigma^2 R^{-6}$ term is the dominant term most of the time near the final singularity (all other terms are relatively small in each "Kasner epoch"—defined below—and a generic recollapsing closed universe is in such an epoch most of the time).

The term ρ_m is the density of the matter particles, which travel on timelike curves. Examples are electrons, protons, human beings, and black holes. It can be shown that $\rho_m = C/R^3$, where C is constant. The term ρ_r is the density of massless particles, which travel along null curves. It is thus composed of all nongravitational radiation fields: photons and massless neutrinos, for example. It can be shown that $\rho_r = D/R^4$, where D is a constant. (Both of the above expressions of the matter and radiation densities in terms of the scale factor assume that there is no significant conversion of matter into radiation or vice versa. Even if such conversion occurs, neither density can drop off faster than R^{-4}, which is the important point for our purposes.) At present, the ρ_m term is the most important term in (2), so the present-day universe is said to be "matter-dominated."

One can obtain (Tipler, 1976b) some very general constraints on the long-term behavior of the matter and shear terms in equations (2) and (3), valid even beyond the point at which these equations break down. If the spacetime is homogeneous, then in ever-expanding universes (Tipler, 1979b)

$$\lim_{t\to+\infty} \inf t^2\sigma^2 < 3/8 \tag{4}$$

If instead the universe ends in a final singularity, then along a timelike geodesic that terminates in this final singularity at proper time t_f we must have (Tipler, 1977)

$$\lim_{t\to t_f} (t_f - t)^2[8\pi G(T_{ab} - \tfrac{1}{2}g_{ab}T)V^aV^b + \sigma^2] \leq \tfrac{1}{2} \tag{5}$$

Roughly speaking, these inequalities say that the shear cannot drop off slower than $1/t^2$ if the universe expands forever, and the shear and matter energy densities cannot increase faster than $1/t^2$ near the final singularity if it recollapses. Conversely, there is considerable evidence (Barrow and Tipler, 1978; Collins and Hawking, 1973) that as a general rule one can compute $R(t)$ in a given regime from the requirement that the dominant term for that regime in the generalized Friedman equation (2) dies off or grows as $1/t^2$. For example, in a matter-dominated regime, $p_m \propto R^{-3}$ is the dominant term by definition, the rule $R^{-3} \propto t^{-2}$ implies $R(t) \propto t^{2/3}$. For a radiation-dominated regime, $R^{-4} \propto t^{-2}$ gives $R(t) \propto t^{1/2}$. Spatially flat universes ($k=0$) will always be either radiation- or matter-dominated, since the only other term in (2), the shear, is apparently zero near the initial singularity. Universes that are not spatially flat (those universes with $k=\pm 1$) will have regimes where the radiation dominates, the matter dominates, the spatial curvature (the k/R^2 term) dominates, and the shear dominates. If the spatial curvature or R^{-2} shear term dominates—as it will in the far future of an open ($k=-1$) universe—$R^{-2} \propto t^{-2}$ gives $R(t) \propto t$. Although Σ is very small today, it is likely to be nonzero, and so near the final singularity in a closed universe, the R^{-6} shear term will dominate eventually, and in this regime the rule $R^{-6} \propto t^{-2}$ gives $R(t) \propto t^{1/3}$. All of the above functional dependences of $R(t)$ in the various regimes, obtained via the rule, are in fact the correct temporal variation most of the time in spatially homogeneous models. The rule fails to give the correct behavior for $R(t)$ only in the spatial curvature-dominated regimes of closed universes; in particular, where $R(t)$ is near the radius of maximal expansion. In these regimes, $R(t)$ is not a simple power of t.

There are two terms, the R^{-2} shear term and the spatial curvature term k/R^2, either of which could be the dominant term in the far future of open ($k=-1$) universes. In fact, both are important in generic open universes. Whenever $R(t)$ varies as a power of t i.e., whenever $R(t) \propto t^n$, where n is some positive constant, the Hubble parameter $H \equiv R'/R$ will vary as $H \propto 1/t$, whatever the value of the constant n. The shear term will be important unless the distortion $\sigma/H \propto \sigma t$ goes to zero asymptotically; it can be shown (Barrow and Tipler, 1978; Collins and Hawking, 1973) that this does not occur generically, so the shear term remains important in almost all open universes.

Both of the R^{-2} terms are absent in the spatially flat, ever-expanding universes, so the long-term evolution of these universes will be either matter- or radiation-dominated. I shall show in Section 3 that for the most part these universes will be matter-dominated as the universe is now, except for a brief period 10^{30} years in the future.

The generic behavior of closed universes in the R^{-6} shear-dominated regime near the final singularity is particularly interesting, at least in closed, spatially homogeneous universes. We have very little knowledge about the behavior of inhomogeneous closed universes in this regime; I can only hope it is qualitatively similar to the homogeneous case.

In a homogeneous closed universe with topology S^3 (the only closed universe topology I shall consider) the shear measures the rate of change in the distortion of the three-sphere. When the shear σ is identically zero at all times the closed universe is isotropic, which means the proper distance around the universe is the same in all directions. If the shear is nonzero, the proper distance around the universe at any given time depends on the direction in space. In developing a feel for the physical meaning of shear, it is instructive to visualize what shear means for a two-sphere. Imagine an observer standing at a point on the two-sphere—the north pole, say—and looking out long two mutually perpendicular great circles through that point. If the sphere is undistorted, the lengths of the great circles will be the same. If the sphere is distorted into an ellipsoidal figure, the length of one great circle will be longer than that of the other.

Suppose the two-sphere universe is shrinking in area as it goes into a final singularity (where the area is zero). The shear measures the rate of change of the distortion, so a nonzero shear means that as our two-sphere universe gets smaller, the lengths of the great circles change their size at different rates: the universe changes its size differently in different directions. A contracting universe means the area is decreasing, but it is quite possible for the length of one great circle to *increase*, and still the overall area will decrease if the other length decreases even faster.

The behavior of the three-sphere universe is qualitatively the same. In three dimensions there would be three mutually perpendicular great circles. A nonzero shear means these great circles are changing their lengths at different rates. The typical situation is for two of the great circles to get smaller very rapidly while the other gets longer, with the net volume of the universe to decrease. When a recollapsing closed universe is decreasing its volume in this fashion—expanding in one direction while decreasing in the other two directions—we say it is in a *Kasner epoch*. The direction of expansion is *constant* throughout the Kasner epoch.

But the universe generally does not remain in a given Kasner epoch all the way in to the final singularity. Rather, the rate of expansion of the expanding great circle will after a time rapidly decrease to zero, and the rate of contraction of the other two great circles will also rapidly decrease. Then the previously expanding great circle starts to *contract* at a faster and faster rate, and one of the previously contracting great circles begins to expand. We can equally well express this by saying that an overall contracting universe expands in one direction while contracting in the other two. The direction of expansion is constant most of the time, but at certain times it suddenly changes. Each regime in which the direction of expansion is constant defines a given Kasner epoch. To repeat: most of the time the collapsing universe is in a Kasner epoch when $\sigma^2 \propto R^{-6} \propto t^{-2}$ is the dominant term in equation (2). The evolution of a recollapsing closed universe during both the Kasner epochs and during the short periods of transition between Kasner epochs is discussed at length in Ryan and Shepley (1975). For evolution during the Kasner epochs, see Chapter 30 of Misner et al. (1973).

This direction dependence of expansion/contraction means the temperature of the background radiation will depend on direction also. The radiation coming from the expanding direction will be red-shifted, while the radiation coming from the contracting direction will be blue-shifted. The precise direction dependence of the temperature is a rather complicated function of the optical depth of the universe at the time. (The optical depth measures the distance a radiation particle can travel before being absorbed.) Approximate formulas have been obtained by Thorne (1967), Misner (1968), and Matzner (1971). For small optical depths—the expected case near the final singularity—the formula simplifies enormously to

$$T(n) = (T_0/R)/\{[\exp(\beta)]_{ij} n^i n^i\}^{1/2} \tag{6}$$

where n^i is a unit vector in the direction the temperature is measured, $T_0/R(t)$ is the temperature averaged over all directions [we put in a factor $R(t)$ explicitly to indicate that this average temperature scales with the universal scale factor; T_0 is a constant], and the exponential factor is a direct measure of the *ratios* of the proper distances around the universe in the three directions. The variation in temperature with direction will be the same at every point in a homogeneous universe. The temperature difference in different directions is a manifestation of the shear gravitational energy, since it is the shear that generates a nonzero β (we actually have $\sigma_{ab} = \beta'_{ab}$, where the prime denotes the time derivative). The temperature difference can in principle provide an energy source for computation in a closed universe near the final singularity.

The extremely rapid contraction in one direction in a shearing universe can cause the disappearance of the horizons in that direction. This fact will be crucial for computation in closed universes, because horizons are the ultimate barriers to communication in space–time.

A horizon is said to exist if there are regions of space that cannot send light signals to each other. If the regions cannot send light signals, then they cannot send signals of any sort, which means it is impossible for them to communicate. But to determine that regions cannot communicate, it is necessary to know the entire future history of the regions, for it may be that the signals merely take a very long time to traverse the distance between the regions, rather than being completely unable to traverse the distance.

Roger Penrose developed in the 1960s a method to easily visualize the entire future (and past) history of a universe, even if that future and past are infinite! A cosmological model is described by its metric $ds^2 = g_{ab}dx^a dx^b$, which may define an infinite space-time volume. Penrose's idea is to replace the coordinates x^a with new coordinates x'^a such that the points at infinity in the old coordinates are at a finite distance in the new coordinates. Furthermore, the new coordinates must be chosen so that

$$ds^2 = \Omega^2\, ds'^2 \tag{7}$$

where Ω is a function of the new coordinates satisfying various conditions, which are not important for our purposes. Now the metric ds'^2 covers the whole of the space-time represented by ds^2 in a finite range of its coordinates; the possible infinities in space and time in the original coordinates have been transferred into the function Ω. Two metrics ds^2 and ds'^2 are said to be *conformally related* if they satisfy (7). This means for space-times that the causal structures—whether regions in the space-time can communicate, or more precisely whether events can be connected with causal (timelike or null) curves—are exactly the same in the metrics ds^2 and ds'^2. Thus if we are interested in the causal structure of the original metric ds^2, all we have to do is throw away the function Ω and study the metric ds'^2 in a small, finite region, for in this region the causal structures will be exactly the same.

The conformal metrics ds'^2 for a number of key cosmological models have been computed, and the region conformal to the entire original cosmological model can be drawn as a two-dimensional figure called a *Penrose diagram* (or conformal diagram), in which the time dimension and one of the three spatial dimensions appear in the figure. The Penrose diagram for the open and the flat Friedman universes is shown in Figure 1, the Penrose diagram for the closed Friedman universe is shown in Figure 2. and the Penrose diagram for the static Einstein universe is shown in Figure 3.

The causal conventions in Penrose diagrams are the same as in Minkowski diagrams: lines at 45 deg off the vertical are the paths of light rays, timelike curves are those whose tangents are less than 45 deg off the vertical, and spacelike curves are those whose tangents are greater than 45 deg off the vertical. Time increases vertically upward, and the horizontal direction is a space direction.

The boundaries of a Penrose diagram represent what are termed *c-boundaries* of the cosmological models. The *c*-boundaries are composed of the singularities and the points at infinity; the *c*-boundary of a cosmology is the edge of spacetime, the "place" at which space and time begin. By convention, singularities are represented by double lines in Penrose diagrams. As can be seen in Figure 2, the initial and final singularities are the only *c*-boundaries in a closed Friedman universe. An open Friedman

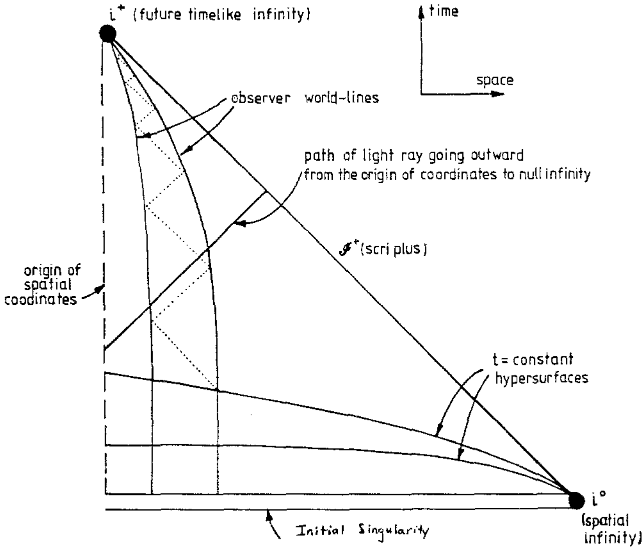

Fig. 1. Penrose diagram for the open and the flat Friedman universes. The horizontal double lines at the bottom of the figure denote the initial singularity. The dashed vertical line denotes the origin of spatial coordinates. For all Penrose diagrams, the conventions for representing timelike, null, and spacelike lines are the same as for Minkowski diagrams. Each point in the diagram represents a 2-sphere except for the world-line of the origin of coordinates, each point of which represents a point. Two observer world-lines are pictured, and those are for observers which are at rest with respect of the background radiation, or equivalently, which have world-lines normal to the foliation of constant mean curvature. All observers who do not accelerate to the speed of light come together in the infinite future at i^+, future timelike infinity. All outgoing light rays hit $\mathscr{I}^+$ (scri plus) at infinite future time. Two leaves (two $t = \text{constant}$ hypersurfaces) of the constant mean curvature foliation are pictured. Each leaf defines a global "now," and each leaf hits i^0 at spatial infinity. The jagged dotted line connecting the two observers denotes light signals being sent back and forth between the two observers. An infinite number of such signals can be exchanged between any "now" and future timelike infinity.

universe, on the other hand, has four distinct c-boundary structures: an initial singularity out of which the entire space-time arose, ∝45-deg line $\mathscr{I}^+$ (called "scri plus") representing "null infinity," which are the points at infinity that light rays (null curves) reach after infinite time, and a single point i^+ which all timelike curves approach for all finite times, and reach after infinite time (with the exception of those timelike curves that accelerate forever and thus approach arbitrarily closely to the speed of light. These curves hit scri plus rather than i^+ at temporal infinity).

The Penrose diagram allows us to define rigorously the notion of "the beginning of time"— the past part of the c-boundary—and the "end of time"— the future part of the c-boundary. Using the c-boundary, we can

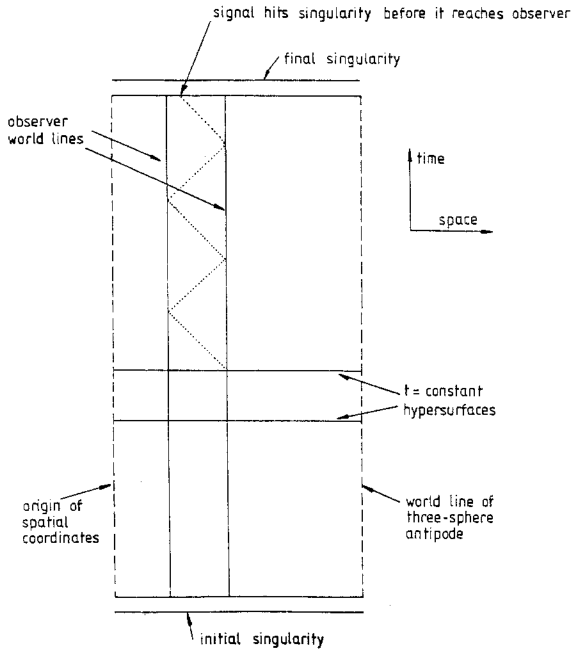

Fig. 2. Pentrose diagram for the closed Friedman universe. The conventions for timelike, null, and spacelike curves are as in Figure 1. The initial and final singularities are denoted by double lines. Each point in the diagram except for those on the dashed lines denotes a 2-sphere. The points on the dashed lines denote single points. The two $t = \text{const}$ hypersurfaces which are pictured denote constant mean hypersurfaces. Each such hypersurface is a 3-sphere. The two dashed vertical lines are thus the world-lines of origin of coordinates and the antipodal points of a 3-sphere. Two observers at rest with respect to the background radiation are shown, and, as in Figure 1, the jagged dotted lines denotes light signals being exchanged between the two observers. In contrast to the case in the open and flat Friedman universes, only a finite number of signals can pass between the two observers before they hit the final singularity. This will be true no matter how close the two observers are.

even define the topology of the "the beginning of time" and "the end of time" in cosmological models. In the closed Friedman universe, the initial and final singularities both have topology S^3, while the initial singularity in the open and flat Friedman universes has topology R^3. In these very special spacetimes, it is even possible to put a metric on the singularities, but in general this will not be possible. It can be shown, however, that if the universe is globally hyperbolic, then there is a natural Hausdorff topology on the c-boundary of the spacetime (Hawking and Ellis, 1973).

Two observers can communicate for all time only if they can send light signals back and forth to one another indefinitely. If two observers lose the ability to send light rays back and forth, we say that an *event horizon* has

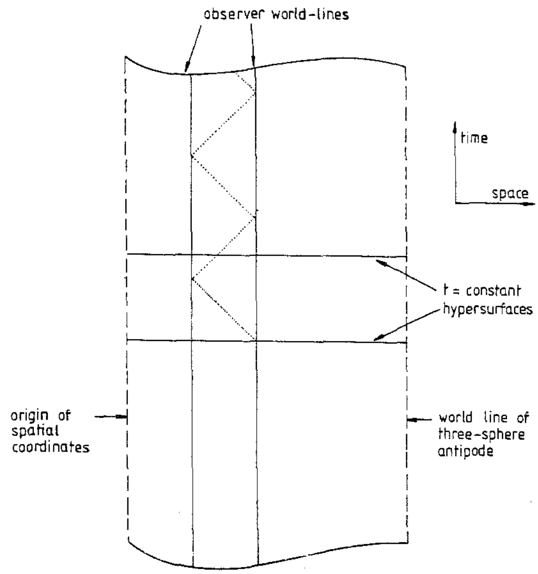

Fig. 3. Quasi-Penrose diagram for the Einstein static universe. The $t = \text{const}$ hypersurfaces are 3-spheres, as in the closed Friedman universe. The curved line at the top and bottom of the figure indicate that the figure continues on indefinitely. The pictured observer world-lines remain equidistant from each other for infinite proper time, so an infinite number of signals can be exchanged between any two observers. Thus the future *c*-boundary of an Einstein static universe is an omega point.

formed between them. It is immediately apparent from the Penrose diagrams that in an open and in a flat Friedman universe, no horizons form and that any two observers, represented by timelike curves, can send an infinite number of light rays back and forth between now and the time when i^+ is reached, because in the Penrose diagram, the timelike curves get closer and closer together as i^+ is approached. In contrast, event horizons *do* form between any two observers in the closed Friedman universe. In Figure 2, the world-lines of comoving observers are shown as vertical lines, and no matter how close the observers are, there will come a time at which it will no longer be possible to connect the two lines with a 45-deg line, which represents a light ray; the final singularity is reached before a light ray from one observer can reach the other. It is simply impossible for computation to continue indefinitely in a closed Friedman universe because it would eventually become impossible for a computer in such a universe even to send signals to different parts of itself!

But not all closed universes have a *c*-boundary structure, or rather a final singularity, like the closed Friedman universe. The Friedman final singularity will occur only when the shear is zero, and as I pointed out earlier in this section, not only is the shear in generic closed universes not zero, it is in fact so large that the evolution of the universe will be *dominated* by the shear near the final singularity. What can happen is that a shear-dominated closed universe can contract so much faster in one direction than a Friedman universe that it becomes possible for light signals to circle the universe in that direction; it is possible to communicate in that direction, and relativists say that the horizon disappears (temporarily) in that direction. Note that it is possible for a horizon to disappear in a given direction and for there still to be an event horizon in that direction. The event horizon disappears also only if it is possible to send signals back and forth in that direction not just once but an infinite number of times. However, if the direction in which the horizon disappears alternately covers all directions, and covers them infinitely many times before the singularity is reached, then it is possible for all observers to send light rays infinitely often back and forth before the singularity is reached. In such a universe there would be no *event* horizons, and there would be no communication barriers to unlimited computation.

Now, two points are defined as distinct in the *c*-boundary only if there are timelike curves that reach these two points and are not contained in the chronological pasts of each other. If all event horizons disappear, then all timelike curves are in the chronological past of each other. Thus the future *c*-boundary of a universe with no event horizons must consist of just a single point; I shall call such a point an *omega point.*

Two simple examples of cosmological models with an omega point are the Einstein static universe (Figure 3) and Löbell space (Hawking and Ellis, 1973). Löbell space is a spacetime constructed by identifying Minkowski space in a certain way to obtain three-torus spacelike hypersurfaces with constant mean extrinsic curvature. These hypersurfaces are Cauchy hypersurfaces for Löbell space. The omega point in Löbell space is a singularity, and it is reached in finite proper time.

In contrast, the omega point in Einstein space is reached only after infinite proper time. It is easy to see from Figure 3 that a light ray can be sent from one observer to another an infinite number of times. Figure 3 gives picture of the causal structure of Einstein space, but it is different from the other Penrose diagrams in that the point at temporal infinity—the omega point—is not brought into a finite distance. A true Penrose diagram for Einstein space, with the omega point brought into a finite distance, has been constructed by Tipler (1986). The future part of this diagram is the same as the future part of the diagram for a closed S^3 universe which begins in an initial singularity like the closed Friedman universe, but approaches the Einstein static universe asymptotically in the future. The Eddington-

Lemaître–Bondi universe is a closed S^3 universe which has just this temporal behavior. The Penrose diagram for an Eddington–Lemaître–Bondi universe is given in Figure 4. As in the open universe, the timelike curves in the Eddington–Lemaître–Bondi universe come closer and closer together as the omega point is reached, so that light rays can pass an infinite number of times between these curves. Penrose diagrams are completely accurate only for spacetimes with spherical symmetry—such symmetry allows two angular coordinates to be suppressed without loss of information—but the causal structure of any closed universe that begins in an initial Friedman-like singularity and ends in an omega point would look qualitatively like Figure 4.

Seifert (1971) has proved that spacetimes in which the future *c*-boundary consists of a single point must have a compact Cauchy surface. That is, cosmologies with an omega point must be closed globally hyperbolic universes. Although open universes like the Friedman universe pictured in

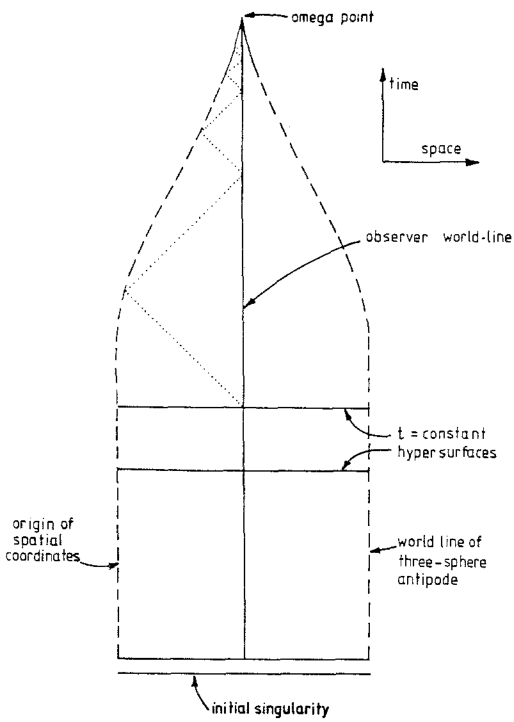

Fig. 4. Pentrose diagram for an Eddington–Lemaître–Bondi universe, a spacetime with an initial singularity and an omega point. It represents a closed universe in which computation is not limited by event horizons, since every observer can communicate with every other observer an infinite number of times.

Figure 1 have a *c*-boundary point i^+ to which all nonaccelerated timelike curves terminate, these spacetimes do not contain an omega point. For scri plus also forms part of the future *c*-boundary, and as I mentioned above, there are timelike curves that terminate on scri plus. Other interesting general restrictions on spacetimes with an omega point have been obtained by Budic and Sachs (1976).

The spatially homogeneous Bianchi type IX model [see Ryan and Shepley (1975)] for an extensive discussion of its properties), which is a shearing closed universe with Cauchy hypersurfaces having topology S^3, was extensively investigated during the late 1960s by relativity groups in the United States, the United Kingdom, and the Soviet Union to see if the horizons in this model disappeared in the past. The conclusion was that it is possible for the horizon to disappear in one direction once, but even this is a rather improbable occurrence in a generic Bianchi type IX model. The probability that horizons will disappear in *all* directions an infinite number of times as the singularity is approached was never actually rigorously calculated, but there are indications that this probability is zero in the vacuum models. Which is to say, there *could* be Bianchi type IX models that have an omega point, but if these models exist, they are of measure zero in the initial data space of the Bianchi type IX vacuum models. Putting in perfect fluids does not change this conclusion, but it is possible that more exotic forms of matter could result in an omega point being more likely. I shall assume that a solution to the Einstein equations with an omega point exists.

The Bianchi type IX models were considered only as models of the *past* singularity. But the evidence is strong (Barrow and Matzner, 1977) that the initial singularity was probably shear-free, so the closed Friedman universe with its regular S^3 topology singularity is a more accurate model of the past than the shearing Bianchi type IX. However, the *final* singularity is by contrast almost certain to be dominated by shear, and so the appropriate use of the Bianchi type IX model is as a model of the actual universe near the *final* singularity, not the initial singularity.

In the homogeneous Bianchi models, the final singularity is all-encompassing. It is occasionally suggested that the final singularity in an inhomogeneous closed universe need not be all-encompassing, but a globally hyperbolic closed universe that begins to recollapse *must* terminate in all-encompassing final singularity. In other words, every observer would end in this final singularity, as stated in Theorem 2 below. [The hypersurface of maximal expansion of a universe that begins to recollapse is called a "maximal hypersurface." A *maximal hypersurface* is defined to be a hypersurface with constant mean curvature everywhere equal to zero. Theorem 2 is proved in Masden and Tipler (1980).]

Theorem 2. If a closed universe obeying the strong energy condition contains a maximal hypersurface, and if on this hypersurface (1) is a strict inequality when V^a is the normal to the hypersurface, then every timelike curve has proper time length less than C, where C is a universal constant.

3. THE EVOLUTION OF MATTER IN THE FAR FUTURE

At the present time stellar births are still occurring, but rates are decreasing exponentially, with a half-life of about 5×10^9 years, due to exhaustion of primordial hydrogen and dissipation of gas from the galaxy (Trimble, 1975). In 10^{12} years, star formation will have ceased. Galaxies will become redder, as the hotter, more massive stars leave the main sequence. The later M-type stars will exhaust their hydrogen cores and also leave the main sequence in about 10^{14} years. Thus after about 10^{14} years, stars will cease to be available energy sources for computation.

The dynamical evolution of stellar and galactic systems in the period 10^{15}-10^{25} years hence will be dominated by decay via emission of gravitational radiation and evaporation of the system's subcomponents (Barrow and Tipler, 1978; Dyson, 1979; Islam, 1977, 1979). The latter process gives the shorter time scales. For example, the average time required to detach a planet from a star by a close encounter with a second star is given by

$$T_N=(nV\sigma)^{-1} \tag{8}$$

where n is the number density of stars, V is the average relative velocity of two stars, and σ is the cross section for an encounter resulting in detachment. A rough guess for the cross section would be $\sigma=2\pi r^2$, where r is the distance of closest approach. The Earth and the other planets would probably be detached if another star went between us and the sun, so for detachment of our solar system's planets, I take $\sigma\approx2\times10^{16}\ \text{km}^2$. In the vicinity of our solar system, we have at present $n=3\times10^{-41}\ \text{km}^{-3}$ and $V=50\ \text{km/sec}$, which gives

$$T_N=10^{15}\ \text{years} \tag{9}$$

Serious disturbance of the solar system will result from encounters on a time scale shorter than (9).

Close stellar encounters will also result in the escape of some stars from the galaxy, since some encounters would result in some stars attaining escape velocity. The details of such losses are exceedingly complex because of subtle relaxation effects (Saslaw, 1973), but the time scale will be closely related to the relaxation time for gravitational encounters (Saslaw, 1973, 1974; Chandrasekhar, 1960; Spitzer and Hära (1958). A very rough estimate of the time scale for evaporation of stars from stellar systems has been given by Dyson (1979). In a system of N stars of mass M in a volume of radius R, the root mean square velocity of the stars will be of the order of

$$V=(GNM/R)^{1/2} \tag{10}$$

The cross section for a close encounter between stars in this system is

$$\sigma=(GM/V^2)^2=(R/N)^2 \tag{11}$$

We obtain the average time between close encounters by inserting (10) into (8):

$$T_N=(nV\sigma)^{-1}=(NR^3/GM)^{1/2} \tag{12}$$

For a typical galaxy, we will have $N=10^{11}$ and $R=3\times10^{17}$ km, so

$$T=10^{19}\text{ years} \tag{13}$$

With the above numbers, the cross section (11) corresponds to a closest approach of about 10^6 km, which is much closer than required to disrupt a solar system. (The stars involved in the collision actually collide.) The time scale for the dynamical relaxation of the stellar system is

$$T_R=T/\log N \tag{14}$$

or $T_R=10^{18}$ years for a typical galaxy. Using the same formulas, we can calculate that clusters of galaxies will evaporate galaxy-size objects in $\sim10^{11}$ years and stellar-sized ones in $\sim10^{23}$ years. The evaporation of objects from the system will leave the system's total energy more negative than before, since the objects leaving the system will almost always have positive energy. The system will thus become more tighly bound as time goes on.

Another mechanism which leads to positive energy loss to the system is gravitational radiation. A mass orbiting around a fixed center with velocity V, period P, and kinetic energy E will lose energy by gravitational radiation at the rate (Misner *et al.*, 1973; Landau and Lifschitz, 1975)

$$dE/dt=V^5(E/P) \tag{15}$$

where units are such that all velocities are measured in units of the velocity of light c, and $G=1$. Thus the time scale for orbital decay via gravitational radiation emission is

$$T=E/(dE/dt)=V^{-5}P \tag{16}$$

For the Earth orbiting the sun, (16) gives $T=10^{20}$ years. For our sun's orbit around the galactic center, with $V=200$ km/sec and $P=2\times10^8$ years, the time scale is 10^{24} years. We can use (16) to obtain the time scale for the emission of all rotational energy of any bound system by equating gravitational energy M^2R^{-1} with the rotational energy to get P and V. This gives a lifetime of Barrow and Tipler (1978)

$$T_{\text{grav}}\sim G^{-3}R^4M^{-3} \tag{17}$$

where I have reinserted the gravitational constant G to show its power dependence. The time scale for a large $10^{15}\ M_\odot$ cluster to radiate away all its energy is $\sim 10^{19}$ years.

The final state after the objects are evaporated and rotational energy has been radiated away is probably a black hole. Dead stars with mass exceeding the Landau-Chandrasekhar limit, $M_{\rm LC} \sim G^{3/2} m_N^{-2} \sim 1.5 M_\odot$, will be the first objects to become black holes, but galaxies and the largest bound configurations—these typically are of order $10^{17} M_\odot$—will eventually follow them on the above time scales.

If the universe recollapses, there will be another period when the above effects will be important: that period over which the universe contracts from 10^{25} years before the final singularity to 10^5–10^{10} years before the final singularity. The lower bound is difficult to predict, because the shear—long-wavelength gravitational waves and curvature anisotropy—will play a dominant role in the evolution of the final state (since $\sigma^2 \propto R^{-6}$) and just at what point the shear will dominate the evolution depends in a very complicated way on the exchange of energy between the R^{-6} and the R^{-2} shear terms in equation (3). However, such exchange will be significant in a closed universe only if the universe is very long-lived [with density parameter $\Omega_0 - 1 \approx 10^{-6}$ or less; see Misner *et al.* (1973) or any other elementary cosmology text for the definition of the density parameter. If the density parameter $\Omega_0 = 1$, the universe is flat; if $\Omega_0 < 1$, it is open; and if $\Omega_0 > 1$, it is closed.] The classical evolution of a closed universe in the case when $\Omega_0 - 1 \approx O(1)$ has been discussed by Rees (1969). When Ω_0 is this large, stars will survive until they are disrupted near the final singularity. The cosmological background radiation will equal the surface temperature of stars when $t \approx 10^5$ years before the final singularity. A star will be disrupted when the background radiation equals the temperature of the star's central region at $t \approx 10^{-1}$ years before the final singularity. Stellar collisions are never a significant factor in stellar disruption. Radiation or neutrino pressure will tend to damp out inhomogeneities on scales shorter than the Jeans length, which, if $\Omega_0 - 1 \approx O(1)$, corresponds to a mass of $\sim M_\odot$. In such a short-lived closed universe, the stars have insufficient time to contribute significantly to the entropy of the universe, so the temperature near the final singularity will go as $T = [R_{\rm now}/R(t)]$ (3 K), where $R_{\rm now} \approx 2 \times 10^{10}$ light-years is the present-day value of the scale factor. The important time scales near the final singularity of a short-lived closed universe are summarized in Table I.

In very long-lived universes, it is possible for stellar radiation to make a significant contribution to the universal entropy and energy density, because in such universes the radiation can be emitted at times when the big bang radiation has been redshifted to very low temperatures: a stellar

Table I. Time Scales in Small, Closed Universe near Final Singularity[a]

Event	Universal scale factor $R(t)$	Temperature (K)
Galaxies merge	$10^{-2}R_{\text{now}}$	300
Sky as bright as the sun's surface	$10^{-3}R_{\text{now}}$	3000
Sky as hot as stellar cores; stars explode	$10^{-6}R_{\text{now}}$	3×10^{6}
Nuclei disintegrate into neutrons and protons	$10^{-9}R_{\text{now}}$	3×10^{9}
Protons and neutrons become free quarks	$10^{-12}R_{\text{now}}$	3×10^{12}

[a]This table applies only to *small* closed universe, i.e., closed universes that begin to contract less than 10^{11} years after the initial singularity. For much larger closed universes, those that do not begin to contract until 10^{30} years after the initial singularity, the temperature near the final singularity will increase as $1/R(t)$; but only the elementary particles e^+, e^-, $\bar{\nu}$, ν, and γ will exist to be heated. Furthermore, the additional radiation from stars at late times will change the constants in the temperature formula $T=(3\text{ K})R_{\text{now}}/R(t)$. Closed universes intermediate in size will have a mixture of dead stars, black holes, and gas which will be heated. R_{now} is the value of the universal scale factor at the present time.

photon in this environment will make a very large contribution to the entropy when it is thermalized near the final singularity.

The time scales so far considered have been governed entirely classical mechanics, including general relativity. In the very long run, the important time scales arise from the decays, due to quantum effects, of various matter structures.

The most important decay both in terms of its cosmological consequences and in terms of its significance for life is proton decay. For example, if the SU(5) GUT is correct, the proton will disintegrate into leptons and photons. There are a number of decay branches via various short-lived particles, but the end result is

$$\begin{aligned} p &\to e^+ + \nu + \bar{\nu} + \gamma \\ n &\to e^+ + e^- + \nu + \bar{\nu} + \gamma \end{aligned} \tag{18}$$

Depending on the decay mode, there will be different numbers of the four particles on the rhs of (18), subject to the conservation of $B-L$, where B and L are the baryon and lepton numbers, respectively, and conservation of electric charge.

The proton lifetime in the $SU(5)$ GUT is 10^{30} years, and a number of other GUTs give a similar lifetime. Experiments have to date failed to detect the predicted proton decay, so it may be that the predictions are wrong. Nevertheless, if quarks and leptons truly lie in the same multiplet in a unified field theory of some sort, then it is very likely there will be transitions between various levels of the multiplet, and this will cause proton decay, even if the lifetime is longer than 10^{30} years. I shall therefore assume that proton decay occurs, and I shall use the 10^{30} year time scale of $SU(5)$. If proton decay occurs via a different process with a quite longer time scale, the qualitative features will nevertheless be the same: the overall decay reaction will still be (18), and the thermal and gravitational effects of proton decay on macroscopic bodies will be the same except that temperatures and evolution rates will have to be scaled appropriately.

Proton decay provides an energy source for large bodies—dead planets, black dwarfs, and neutron stars—which will prevent them from cooling to the temperature of the radiation background. Since these bodies are effectively electrically neutral, the positrons produced in the reactions (18) will be immediately annihilated, so in such bodies the net effect of (18) is to turn matter into energy. The energy released in proton decay will keep neutron stars at a temperature of 100 K, and black dwarfs and Earth-sized planets at 5 and 0.16 K, respectively, for around 10^{30} years into the future (Feinberg, 1981; Dicus *et al.*, 1982, 1983). These numbers are calculated by equating the usual cooling law,

$$dE/dt = 4\pi R^2 \sigma_{SB} T^4 \tag{19}$$

where σ_{SB} is the Stefan–Boltzmann constant, with the power generated by proton decay:

$$\begin{aligned} dE/dt &= (1\ \text{GeV/proton})(\#\text{protons/proton lifetime}) \\ &= (6 \times 10^{16}\ \text{ergs/sec})(M/M_\odot) \end{aligned} \tag{20}$$

White dwarfs cool to black dwarfs at 1 K in 10^{20} years in the absence of proton decay, but proton decay will deep them at 5 K during the period $10^{17} < t < 10^{30}$ years. At the end of 10^{31} years only about 5×10^{-5} of the original mass of the star or planet will remain: planets will have become asteroids and black dwarfs Earth-sized planets, and the process will continue until the mass has been entirely converted into energy. By about 10^{32} years, the most massive solid structures, which have a mass of about $10M_\odot$, will have completely disappeared.

As implied by Frautschi (1982), proton decay spells the ultimate end for computers based on protons and neutrons. Baryons are disappearing at the exponential rate $N(t)\exp(-t/10^{30}\ \text{years})$, where $N(t)$ is the number of protons in the structure under consideration, which may be increasing. Even if the size of a computer were to expand its volume at the speed of light, the number of protons in that volume would increase only as $\underline{N}t^2$, where the constant $\underline{N}$ is bounded above by the cosmological baryon number density today, so the maximum value of $N(t)$ would be $\underline{N}t^3$. The exponential decrease will eventually defeat the power law increase, and baryon-based computers will disappear if the universe is flat or open, or if it is a sufficiently long-lived closed cosmology. Setting $\underline{N} = 10^{100}$ and $\underline{N}t^3 \exp(-t/10^{30}$ years) = 1 gives $t = 10^{33}$ years for the time by which *all* atom-based computation must cease. More generally, if the proton lifetime is t_p, all atom-based computers will have disintegrated by $10^3 t_p$ years.

The conversion of mass into energy via proton decay can have dynamical effects on cosmological evolution, for it can change a matter-dominated universe into a radiation-dominated one. The cosmological effects of proton decay have been investigated by Barrow and Tipler (1978), Page and McKee (1981, 1983), and Dicus, *et al.* (1982, 1983). All of these authors agree that if we ignore the possibility of exotic forms of matter, then in all Friedman universes, the only matter remaining after 10^{33} years is an electron/positron plasma, which originates enitrely from the protons that did *not* form clumpy matter—stars, planets, asteroids, rocks, dust particles, or any bound group of atoms. When protons decay in clumpy matter, the electrons and positrons annihilate, as pointed out above, and so cannot contribute to the plasma. About 1% of the matter will be in the form of atomic hydrogen beyond 10^{20} years, so these atoms will be the source of the electron/positron plasma. Dicus *et al.* point out that in an open universe, there will be a brief period between t_p and $10^3 t_p$ in which the exponential decay of the protons will generate radiation so rapidly from the matter that the universe will be radiation-dominated. After that time the matter density of the electron/positron plasma will dominate because its density falls off as R^{-3} while the radiation density falls off as R^{-4}. All of the above authors agree that in an open universe, the cosmological expansion will be too rapid for the electrons and positrons in the plasma to recombine into positronium, at least via electrical forces, though Page and McKee raise the possibility that gravitational clumping could cause the electrons and positrons to recombine. This seems rather doubtful to me because the gravitational and electrical forces are both R^{-2} laws in the distance regime in question. But like particles repel electrically, while gravity is always attractive, and, as Page and McKee point out, this difference could lead to clumping when many-body interactions are properly taken into account.

In summary, taking into account the classical time scales discussed earlier in this section, the matter in the universe at 10^{30} years will consist of 90% dead stars and planets being maintaned at a temperature between 100 and 0.1 K by proton decay; 9% galactic mass size black holes from the evapoaration and collapse of galaxies; and 1% atomic hydrogen. All of this material will be immersed in a radiation bath of photons and neutrinos, whose density relative to matter is increasing due to proton decay. Between

10^{30} and 10^{33} years the dead stars and planets will disappear, leaving the black holes, an electron/positron plasma, and radiation. The radiation density will dominate both the plasma density and the black hole density. After 10^{33} years, the radiation density will have decreased sufficiently far so that the black holes will be the dominant component of the universe's mass density.

But black holes do not last forever, any more than protons do. Hawking (1974, 1975) has shown that quantum effects cause black holes to radiate away their mass, with the mass being entirely converted into radiation at the end of $10^{66}(M/M_\odot)^3$ years. Thus galactic mass black holes ($10^{11}M_\odot$) will disappear in 10^{99} years, and supercluster mass black holes ($10^{17}M_\odot$) will disappear in 10^{117} years. Barrow and Tipler (1978) argue that the evaporation of black holes combined with the expansion of the universe will be sufficiently rapid to overcome the increase of black hole mass due to black hole coalescence induced by gravitational attraction. Page and McKee (1981) regard this question as still open, because of the many-body attraction mentioned above. If Barrow and Tipler are correct, supercluster mass black holes will be the most massive black holes ever to form. If so, after 10^{108} years the matter in the universe will consist entirely of an electron/positron plasma in a radiation bath of neutrinos and photons, as Barrow and Tipler were the first to point out.

Both Barrow and Tipler, and Page and McKee, agree that in a flat ($k=0$) and in a long-lived closed universe, the rate of expansion of the universe will be sufficiently slow so that almost all of the electrons and positrons in the plasma will recombine. The particles will recombine into positronium when the total energy of an electron–positron system becomes negative. The only energies the particles have in a flat universe are the Coulomb energy of attraction and the random thermal energy of motion. The thermal energy of the electrons and positrons comes from the energy of proton decay. The rms initial momentum P of the electron or positron produced in a proton decay will be $P=\gamma m_e$, where m_e is the mass of the electron (and $c=1$); probably $\gamma \approx m_p/2m_p \approx 10^3$, where m_p is the proton mass. This initial momentum will redshift as $\gamma m_e t_p^n R^{-1}$, where the constant n is defined by $R(t) \propto t^n$. Thus the thermal kinetic energy will scale with the expansion of the universe as $E_K \approx P^2/m_e \approx \gamma^2 m_e t_p^{2n} R^{-2}$. If at t_p the fraction of the mass in $e^\pm$ is f_e, the number density N of the $e^\pm$ will decrease because of the expansion of the universe as $N \sim f_e m_e^{-1} R^{-3}$, and the average distance between e will grow as $r \sim N^{-1/3} \sim f_e^{-1/3} m_e^{1/3} R$. The sum of thermal energy and Coulomb energy is thus

$$\gamma^2 m_e t_p^{2n} R^{-2} - e^2 f_e^{1/3} R^{-1} \tag{21}$$

If we assume a matter-dominated flat universe where $R(t) \propto t^{2/3}$, (21) will go negative at

$$t_b \sim (10^{78}\text{ years}) f_e^{-1/2} (t_p/10^{30}\text{ years})^2 \tag{22}$$

This is the time when most positronium will be formed by two-body collisions in a flat universe. It will occur somewhat earlier in a closed universe because $R(t)$ is not increasing quite as fast as $t^{2/3}$, and because the sum of cosmological expansion and binding energies, which is negative, must be added in the closed universe case to (21).

However, Page and McKee have shown that three-body collisions of the form

$$e^+ + e^- + e^\pm \rightarrow Ps_n + e^\pm \tag{23}$$

where Ps_n denotes positronium with principal quantum number n, will actually cause most $e^\pm$ to become bound long before t_b, due to recombination into positronium states that have binding energy much greater than E_K. The true positronium formation time scale is

$$t_{pos} \sim (10^{71}\text{ years}) f_e^{-2/3} (t_p/10^{30}\text{ years})^2 \tag{24}$$

where we have assumed $R(t) \propto t^{2/3}$. The time scale (24) will be smaller than (22) unless $f_e \leqslant 10^{-42}$, which seems highly unlikely (and would contradict my earlier calculations). Thus most of the free electrons and positrons in the plasma will bind around time t_{pos}, going typically into an orbit with principal quantum number a bit below the value

$$n \sim 10^{22} f_e^{-4/9} (t_p/10^{30}\text{ years})^{2/3} \tag{25}$$

These positronium states have radii of

$$r_n \sim (10^{12}\text{ megaparsecs}) f_e^{-8/9} (t_p/10^{30}\text{ years})^{4/3} \tag{26}$$

which is much larger than the radius of the visible universe today. In this state, the orbital velocities of the electron and positron about each other are about 10^{-4} cm/century.

The state Ps_n will gradually decay by emission of photons to the ground state, where the positronium will rapidly annihilate. Page and McKee used the classical power loss formula for electromagnetic radiation from a dipole to calculate the decay time:

$$\begin{aligned} t_{decay} &\approx E_n/(dE/dt) \approx 2m_e^{-1} e^{-10} n^3 l^5 (n+l)^{-2} \\ &\approx m^{-1} e^{-10} n^6 \end{aligned} \tag{27}$$

For comparison, Bethe and Salpeter (1957) give exact times for decay from the singlet and triplet S states as $1.25 \times 10^{-10} n^3$ and $1.4 \times 10^{-7} n^3$, respectively, and Sakurai (1967) gives a decay time proportional to $n^{4.5}$, so the decay time (27) will apply only if the positronium forms in an $n = l$ state most

Table II. Time Scales in Open, Flat, and Very Large Closed Universes[a]

Event	Time scale (years)
Sun leaves Main Sequence	5×10^9
Large clusters evaporate galaxies	10^{11}
Stars cease to form	10^{12}
Longest lived stars use all their fuel, and cool to very low temperature	10^{14}
Dead planets detached from dead stars via stellar collisions	10^{15}
Dead stars evaporate from galaxies (approximately 90–99% of stars will evaporate; 1–10% will collect in galactic centers to form gigantic black holes)	10^{19}
Neutron stars cool to 100 K	10^{19}
Orbits of planets decay via gravitational radiation	10^{20}
Dead stars evapoarate from galactic clusters (stars are at 100 K due to proton decay)	10^{23}
At this stage matter consists of about 90% dead stars, 9% black holes, and 1% Atomic hydrogen and helium	
Protons decay [according to $SU(5)$ GUT]	10^{30}
Stars evaporate via proton decay (GUT)	10^{33}
Most matter in the universe is now in the form of e^+, e^-, $\bar{\nu}$, ν, γ	
Solar-mass black holes evapoarte via Hawking process	10^{64}
Ordinary matter liquifies due to quantum tunneling	10^{65}
In flat and closed universes, most e^+ and e^- form positronium (in open universes, most e^+, e^- remain free)	10^{71}
In flat and closed universes, positronium decays via cascade, releasing 10^{22} photons	10^{116}
Supercluster-mass black holes ($10^{17}M_\odot$) evaporate via Hawking process	10^{117}
Protons decay via Hawking process	10^{128}
If ordinary matter survives decay via GUTs or Hawking process, it decays to iron	10^{1500}
All iron collapses into black holes	$10^{10^{26}}$

[a]This table is a complete list of all significant time scales for the evolution of matter. However, it should be noted that some processes listed will preclude the operation of others. For example, if all protons decay via GUTs at 10^{30} years, there will be none remaining to decay via the Hawking process at 10^{128} years. In view of our ignorance concerning the operation of some of these processes (the predicted GUT decay has not been seen experimentally, and may not exist), it is best to list *all* possible processes, and point out that the exact evolutionary sequence is unknown.

of the time (as seems reasonable). The typical transition will be $\Delta n = 1$, so something of the order of 10^{22} photons will be generated as the Ps_N state cascades downward. Putting (25) into (27) gives the typical time of annihilation:

$$t_{\text{decay}} \approx (10^{116} \text{ years}) f_e^{-8/3} (t_p / 10^{30} \text{ years})^4 \qquad (28)$$

The fraction of free electrons and positrons decreases rapidly in flat universes, but they bind in such high quantum numbers that by the time they have decayed the radiation bath will be redshifted to such low mass densities that a flat universe will be *always* matter-dominated by the electrons and positrons.

The evolution of the photon spectrum is complex, due to the sequence of first baryon and then positronium decays. Page and McKee have given some estimates for the evolution of the spectrum, the main conclusion being that the energy density of the photons arising from positronium cascades and annihilation completely dominates all other contributions to the radiation background if in fact the Barrow–Tipler assumption that black holes larger than supercluster mass never form. If this is not true, the radiation background may become dominated by emission from black holes. (If black holes do not have an upper bound in mass, then the assumption of homogeneity that everyone takes will also be unwarranted.)

Thus after 10^{116} years, the matter of the universe consists of an electron–positron plasma, with a good percentage of positronium in the flat and long-lived closed universes immersed in a radiation bath fed by decays of the positronium. Essentially no black holes, no stars, no planets, or other solid material remains.

If proton decay does not occur by GUTs, it is likely that protons will decay via the Hawking process. Time scales for this decay scenario are most uncertain, but the latest calculations give a proton lifetime of 10^{128} years. If protons survive this long, the summary in the preceding paragraph would be the same, except the number 10^{116} would be replaced by 10^{128}, and the radiation bath would be fed by proton decay.

If proton decay does not occur at all, then Dyson has shown that quantum tunneling will cause the eternal matter to decay first to iron (time scale 10^{1500} years) and the iron would collapse to black holes in time scales of $10^{10^{26}}$ years. All of the matter time scales are collected together in Table II.

4. THE COSMOLOGICAL LIMITS ON COMPUTATION

Computation in the far future was first discussed briefly in papers by Dyson (1979) and Frautschi (1982). A program can in principle be run on many types of hardware, and even in the far future of a flat Friedman universe matter in the form of electrons, positrons, and radiation will continue to exist. The basic problem of physical computation is to determine if the forms of matter that, as I outlined in Section 3, will exist in the far future can be used as construction materials for computer elements, if there is sufficient energy in the future environment to run the programs, and if the causal structure of the universe will permit unlimited communication between various parts of the computer.

If computation is to be unlimited in the far future of the universe—if the construction of a true universal Turing machine is to be an actual possibility—the following three conditions must hold:

1. Information processing must be able to continue along at least some future-endless timelike curve γ all the way to the future c-boundary of the universe.
2. The amount of new information *processed* in $J^-(\gamma)$ between now and the c-boundary is infinite.
3. The amount of information *stored* in $J^-(\gamma) \cap S(t)$, where $S(t)$ denotes the constant mean curvature foliation of the universe, diverges as the leaves of the foliation approach the future c-boundary.

The global instant "now" is defined to be all those events contained in the leaf of the constant mean curvature foliation that passes through the Earth at the present time. This definition assumes that a constant mean curvature foliation exists, but the definition can be generalized to apply in other spacetimes. However, such a generalization will not be necessary, for I have indicated in Section 2 that a physically realistic cosmology will have a constant mean curvature foliation.

I have required the stored information to grow in the causal past $J^-(\gamma)$ of a single timelike curve because events must be in the causal past of a timelike curve if they are to be able to communicate with the curve. Computer elements located outside $J^-(\gamma)$ cannot communicate with the element that moves along the timelike curve γ, and so cannot be regarded as part of the same computer. [Since $J^-(S)$ is the closure of $I^-(S)$ for any set S in a globally hyperbolic spacetime (Hawking and Ellis, 1973), I could replace $J^-(\gamma)$ with $I^-(\gamma)$ in the above conditions without changing the definition significantly.] The definition does not preclude computer elements existing elsewhere, but the information generated by other computer elements counts only if the other elements can eventually communicate with the elements around the curve γ.

It is extremely important to note that there need be no correspondence between the duration of various measures of physical times, such as proper time, and the number of bits processed in that time interval. It is quite possible for the universe to exist for only a finite proper time in the future before ending in the c-boundary—as happens in closed universes—and yet for an infinite number of bits to be processed in that time interval. All that is required for this to occur is for the rate of information processing as measured in proper time to diverge sufficiently rapidly as the final singularity is approached. I would claim that the appropriate measure of time duration by computers in a given environment is not in general proper time but the length of time it takes to process a bit. In the current physical (and biological) environment, the bit processing rate is directly proportional to proper time, and this is why we consider that proper time measures time correctly. But if the bit processing rate of a computer in an environment were increasing relative to proper time, it would be appropriate to reject proper time as the computer clock time. The appropriate measure of *physical* time, and the fact that this measure may not be the same as proper time in a given cosmological epoch, has been discussed at length by Misner (1969).

But it is not sufficient for the number of bits processed to be infinite for a true universal Turing machine is to be physically possible. If a computer with a finite amount of information storage—a finite state machine—were to operate forever, it would start to repeat itself. A universal Turing machine is an infinite state machine (though "potentially infinite" would be a more appropriate nomenclature), and an infinite state machine is possible only if condition 3 holds.

The absolute minimum amount of energy required to read a given amount of information is determined by second law of thermodynamics. If ΔI is the information read in bits, then the second law requires

$$\Delta I \leq \Delta E / kT \ln 2 = (\Delta E / T)(\text{ergs/K})(1.05 \times 10^{16}) \tag{29}$$

where k is Boltzmann's constant, T is the absolute temperature in degrees K, and ΔE is the amount of free energy expended. The inequality (29) is due to Brillouin (1962). If the temperature at which the computer is operating is higher than absolute zero, there is a minimum amount of energy that must be expended to read a bit of information.

In the present cosmological epoch, the lowest temperature that physics will effectively permit computers to operate is the temperature of the background radiation, which is 3 K. If we put this temperature into (29), then the total amount of information that could be read in the present epoch by using the entire mass-energy of the Earth is $\Delta I \leq 10^{64}$ bits. For the mass-energy of the entire solar system, we would have $\Delta I \leq 10^{70}$ bits; for the entire galaxy $\Delta I \leq 10^{81}$ bits; and for the mass-energy in all the matter in the entire visible universe, $\Delta I \leq 10^{98}$ bits.

These upper bounds can be lowered only by a decrease in the cosmological temperature. Since $T/(3\text{ K}) = R_{\text{now}}/R(t) \approx 2 \times 10^{10}$ years$/t$, the cosmological temperature will drop by only a factor of 2 over the next 20 billion years, so the upper bounds on the amount of information that can be read will apply to all computation over this length of time.

If inequality (29) is divided by the time difference Δt and the limit $\Delta t \to 0$ is taken, we obtain a constraint on the information processing rate:

$$(dI/dt) \leq (dE/dt)(1.05 \times 10^{23} \text{ bits/sec W})/T \qquad (30)$$

where as before the temperature is measured in degrees Kelvin. At room temperature (300 K) the thermodynamic limit of computer reading speed per unit power is about 10^{21} bits/sec per W. As of this writing the average off-the-shelf microcomputer works at about 10^{8} bits/sec per W, while state-of-the-art super computers work at 10^{10} bits/sec per W. Even if (29) applied to all forms of information processing, and not just to reading of new information, we would still have a long way to go before reaching the limit (30).

But (29) does not apply to computation *per se*, just to the loading of new information into a computer, and permanently recording the results of the computation. Inequality (29) can be derived from several quite different assumptions. Brillouin (1962) obtained (29) by calculating the minimum amount of energy needed to *measure* one bit of information; in computers, measuring would correspond to *reading* a bit. (If there was no minimum, Maxwell's demon could operate, thereby contradicting the second law.) Von Neumann derived (29) by calculating the minimum amount of energy required for accurate transmission of a bit from one logical gate to the next (Porod *et al.*, 1984a, b). Landauer arrived at (29) by arguing that computation is logically irreversible (Landauer, 1961; Landauer and Woo, 1971); that is, information must be thrown away at each computation step. Both the Brillouin and the von Neumann arguments are founded solidly on the second law as generalized by information theory, but Bennett (1973) has pointed out that Landauer's derivation is defective, for computation is in actuality logically reversible. To make a reversible computer, all one has to do is retain all the bits of information left over from the intermediate steps, and once the computation is completed, run the computer in reverse. This will restore the computer to its original state, with an arbitrarily small amount of energy dissipated. A number of ideal physical models of reversible computers have been published (Toffoli et al., 1982; Zurek, 1984). The only time that energy must be dissipated is when information must be thrown away, which is when the program is initially read, and when the final result of the computation is recorded. To both of these processes the inequality (29) applies, though even in these cases one could in principle record all possibilities of a string of symbols the length of the input and the output, which would obviate the necessity of throwing away any information.

However, the number of memory elements required to record all possibilities would increase exponentially with the length of the input. In order to avoid losing the information recorded to thermal fluctuations, the two states of each memory element (necessary to record one bit of information) would have to differ in energy by kT. Now the synthesis of such a memory element would itself require an amount of energy of order kT or greater, for the very material to construct the computer element would have to be located, and this act of locating is equivalent to the measurement that Brillouin showed requires expenditure of energy given by (29). Thus, retaining all possibilities in memory would cost more in energy than throwing away information. Furthermore, Bennett (1982), following von Neumann (see Porod et al., 1984a, b), has pointed out that preventing the information used in intermediate steps in the computation process from being destroyed by thermal fluctuations may require net energy dissipation of order kT per step. In fact, Bennett shows that the transcription process in DNA replication dissipates energy at a rate of about two orders of magnitude above kT per step, and he suggests this is due to the requirement of information stability under thermal fluctuations. (The models of reversible computation mentioned above are apparently unstable to thermal fluctuations.) Furthermore, I shall show below that the transmission of information from one part of the computer to another on cosmological length scales is likely to require the order of kT per bit. I shall therefore assume that (29) places a fundamental limit to information growth in the far future.

From (30) we have the following inequality between the total information processed in the future and the energy required to process it:

$$I = \int_{t_{\text{now}}}^{t_{c\text{-bound}}} (dI/dt)\, dt \leq (k \ln 2)^{-1} \int_{t_{\text{now}}}^{t_{c\text{-bound}}} T^{-1}(dE/dt)\, dt \tag{31}$$

where the upper bound $t_{c\text{-bound}}$ is the time the c-boundary is reached. The value of the integrals in (31) do not depend on which measure of time duration is used.

By condition 2, the leftmost integral must diverge if information processing and storage is to be unlimited, which implies that the rightmost integral must also diverge. In an open or flat cosmology, it is possible for the rightmost integral to diverge even if the total energy used,

$$E = \int_{t_{\text{now}}}^{t_{c\text{-bound}}} (dE/dt)\, dt \tag{32}$$

is finite. Since the temperature goes to zero as the c-boundary is approached in these cosmologies, the information processed can diverge with the total energy being used remaining finite if the information is processed sufficiently slowly. In closed universes the integral (32) must diverge, and diverge very rapidly near the final singularity, since the temperature diverges as $T \sim 1/R(t)$. It shall show that it is possible in principle for the rightmost integral in (31) to diverge in all three basic cosmologies: open, flat, and closed.

What will be the most important energy source in the far future? At present, the most important energy source is matter: mass is converted into energy in stars via thermonuclear fusion, or via radioactive decay of heavy nuclei in bulk matter. But matter is gradually being used up, and no matter how efficient the conversion of energy into information, there are the finite upper bounds that I calculated above to the amount of information that can be read and recorded by the matter available in any finite region over the next 20 billion years. Therefore, computation may use up all the material in the solar system in time scales that are short in comparison with the age of the universe. Thus for computation to continue it will be necessary to expand from the solar system, and gain control of new material. On time scales of tens of billions of years, the total region being used for computation will be an expanding sphere, with almost all of the activity concentrated in a narrow region within a distance ΔR of the surface of the sphere. The interior of the sphere will be an essentially dead region, the matter having been converted into information during the previous eons. The sphere will be expanding on net at some fraction of the speed of light, so on the average the region being used for information processing and storage and the net information stored will be increasing as t^2 (if the interior had not been exhausted, the increase would be proportional to the volume of the sphere rather than its area, or t^3). Thus, although perpetual exponential growth of information stored is not allowed by the laws of physics, a power law growth is allowed. If the average expansion rate, as measured in the local rest frame of inner boundary of the expanding sphere, is always greater than the current Hubble expansion of 50–100 km/sec per megaparsec, then the growth can continue as t^2 for the next 10^{30} years, until the decay of protons becomes important—indefinitely, if we assume protons are forever.

By the end of the period 10^{30}–10^{32} years, the only matter surviving will be electrons and positrons from the decays of single atoms in interstellar space. Frautschi (1982) has considered various possible energy sources, such as Hawking radiation from black holes, and the energy from electron–positron annihilation. He concludes that in open universes, black holes would just barely supply sufficient energy, but the density of electrons and positrons would not. However, it seems to me that neither of these would be the main energy source in the far future.

As I discussed in Section 2, the most important form of energy available in this epoch will be the shear energy, so it is the most probable energy source for computation. As I discussed in Section 2, the shear energy can be extracted by making use of the temperature differential it generates. By

Carnot's theorem, the efficiency of energy extraction should be proportional to $\Delta T/T$, which is independent of the scale factor R by equation (6), so the percentage of energy extracted from the shear energy should be independent of time unless the distortion parameter $\{\exp \beta\}_{ij}$ goes to zero asymptotically, which it cannot do because the shear does not, and $\sigma^2 = \beta'^2$. The shear energy will be equally available at all points inside the sphere of life and not just on the surface as was the matter energy (at least in an approximately homogeneous universe, and I will assume, as observations suggest, that the universe is homogeneous). Thus the region being used for computation can continue its expansion outward, but also begin to reuse the desert of the interior, until the region being used for computation is growing proportionally to the volume rather than just the surface area. The total energy available to the whole computer will in the long run be

$$E \sim \rho R^3 \tag{33}$$

where ρ is the energy density of the available energy source, which in the end will be the shear. For open universes we have $R \sim t$ and $\sigma^2 \sim \rho \sim t^{-2}$, so $dE/dt \sim \text{const}$, where t is the proper time. Putting these relations and (33) into (31), remembering that $T \sim 1/R \sim 1/t$, and absorbing all constants into one constant C, we get

$$I \leq C \int t\, dt \sim t^2 \tag{34}$$

which *diverges* as $t \to +\infty$. For flat universes, the only energy source is the electron–positron plasma, so $\rho \sim R^{-3}$ neglecting annihilation. Thus, neglecting annihilation, we have $E \propto \text{const}$. If this finite amount of energy is used slowly, say on the average $dE/dt \leqslant t^{-\delta}$, then the total energy used over infinite proper time will be finite if $\delta > 1$. We have $R \sim t^{2/3}$ for matter-dominated flat universes, so (31) gives

$$I \leqslant C \int t^{-\delta}(t^{2/3})\, dt \sim t^{-\delta+5/3} \tag{35}$$

which *diverges* as $t \to +\infty$ if $\delta < 5/3$. Thus, if the energy use for all purposes together with the particle annihilation is slow enough, the amount of information processed can diverge in flat universes.

For closed universes, the R^{-6} shear term will eventually dominate. But before that epoch is reached, the universe, if it is closed, will pass through epochs of matter, radiation, R^{-2} shear, and curvature domination, the sequence and duration of each epoch depending on the lifetime of the closed universe. The nature of information processing in the late expanding and early contracting periods will be essentially the same as its predicted nature in the expanding periods in open and flat universes, which I described at length above. I shall therefore consider only the R^{-6} shear-dominated epoch of a closed universe, for it will be in this epoch that the information integral will be divergent or convergent.

The energy available for computation near the final singularity will be, according to (33),

$$E \sim (\Sigma^2/R^6)(R^3) \sim \Sigma^2 R^{-3} \sim \Sigma^2 t^{-1} \tag{36}$$

so $dE/dt \sim t^{-2}$, where t is the proper time before the final singularity is reached at $t=0$. Since $T^{-1} \sim R \sim t^{1/3}$, the right-hand integral in (31) gives

$$I \leqslant C \int (t^{-2})(t^{1/3})\, dt \sim t^{-2/3} \tag{37}$$

which *diverges* as $t \to 0$. Thus, even though the energy used for information processing must diverge very rapidly as the final singularity is approached, we see that there is sufficient energy in the form of shear to provide it.

The ultimate form of energy available for computation is thus gravitational energy in the form of shear. Gravitational energy is actually the ultimate energy source in many circumstances. For example, Zel'dovich and Novikov (1971) have pointed out that since the neutrons in a neutron star (the final state of a type II supernova) are in a *higher* energy nuclear state than the particles, protons and electrons, making up the original star, no net nuclear energy is liberated during the evolution of a massive star. Rather, the energy that goes into the synthesis of the heavy elements ultimately comes from the gravitational potential energy that is liberated as the radius of the core of the star shrinks from several hundred thousand kilometers down to neutron star size of a few kilometers. In Newtonian gravitation, there is no limit in principle to the amount of energy we can extract from a collapsing star, for we can keep shrinking the radius of the star down to zero. However, Schoen and Yau (1978, 1979; see also Witten, 1981; Horowitz and Perry, 1982; Gibbons et al., 1983) have shown that in general relativity, there is such a limit: the initial mass of the star. The reason is that a star can be shrunk to its Schwarzschild radius, and no further. However, the Schoen–Yau theorem assumes that spacetime is asymptotically flat, and that event horizons must form when the gravitational field becomes too intense. In a spacetime with an omega point, there are no horizons, so there are no black holes. Thus, in this type of closed universe, it becomes possible to extract an infinite amount of energy just as in Newtonian gravitation.

As in the case of the open universe, the efficiency of energy extraction for computation near the final singularity will be independent of the scale factor R, but it will be dependent on the distortion parameter β. I pointed out in Section 2 that if communication is to be possible arbitrarily close to the singularity, the horizons must continue to disappear, and this requires β to continue to alternate in size from very small values to very large values. On the average, β will not asymptotically approach zero, so on the average the efficiency of energy extraction will have a lower bound, which we can absorb in the constant C in (37).

In the above evaluations of equation (31) for the open, flat, and closed universes, I have assumed that $T \propto 1/R(t)$, which is the adiabatic variation. In reality, the temperature variation will be nonadiabatic, because in processing information, waste heat is being generated at the rate dE/dt, and this waste heat will raise the temperature of whatever thermal sink is used. If unlimited computation is to be possible, a thermal sink must be found which can absorb heat sufficiently fast so that information processing will not be incinerated by its own waste heat. In Dyson's model (1979), which was concerned with the indefinite survival of life rather than unlimited computation, waste heat elimination was a major difficulty facing life in the far future. Dyson assumed that life was restricted to a constant comoving volume, and that heat was eliminated by radiation to the exterior of the comoving volume.

Radiating waste heat to an exterior region is difficult in open and flat universes. It is absolutely impossible in closed universes in which computation is being carried out over the entire universe, for in such a case there is no exterior region. Therefore, a heat sink copresent with the computer must be used. The obvious choice for such a heat sink is the thermal radiation background. If waste heat for information processing is dumped into the radiation background as it is generated, the energy density of the background will rise at the same rate as the energy density of the energy source, which means $E/V \sim t^{-2} \sim T^4$. This gives $T \sim t^{-1/2}$. For shear-dominated closed universes, this implies that the temperature will rise faster as the universe collapses, as $T \sim R^{-3/2}$ rather than as $T \sim R^{-1}$. For open universes, the temperature will fall off more slowly, as $R^{-1/2}$ rather than as R^{-1}. If these nonadiabatic temperature variations are inserted into (31), we find that $\int T^{-1}(dE/dt)\, dt$ diverges as $t^{-1/2}$ in a shear-dominated closed universe, and as $t^{1/2}$ in an open universe. For flat universes, the integral will still diverge provided $\delta < 5/3$. To summarize: waste heat does not seem to pose a problem for continuing information processing in either the open, flat, or closed universes.

I have argued that event horizons cannot exist if unlimited information processing is to be possible, because such horizons would prevent communication between different computers, and even different parts of the same computer. One might wonder, however, if a single computer could nevertheless process an infinite amount of information in the ever-shrinking region with which it could communicate, by processing information faster than the communication region is shrinking. I can now show that this is impossible on energetic grounds.

At any cosmic time t, the region from which the observer γ can receive signals is $J^{-}(\gamma) \cap S(t)$. In the Friedman universe, the boundary of this region is determined by the ingoing radial null geodesics, which satisfy $ds^2 = 0$. Since the closed Friedman metric is given by

$$ds^2 = -dt^2 + R^2(t)[dr^2 + \sin^2(r)(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

where r is the comoving distance, the null geodesic equation $ds^2 = 0$ gives $dr = -dt(1-r^2)^{1/2}/R(t)$. The proper radius L of the communication region then decreases as $dL = R(t)\, dr(1-r^2)^{-1/2}$, so $dL = -dt$, and thus the proper radius of the communication goes as $L \sim t$ near the final singularity at $t = 0$. In the radiation- or matter-dominated Friedman universe, $R(t) \sim t^{1/2}$ or $R(t) \sim t^{2/3}$, respectively, so the proper volume of the communication region goes as $V \sim L^3 \sim R^6(t)$ or $\sim R^{9/2}(t)$, respectively. The proper energy density can rise only as $R^{-4}(t)$ or $R^{-3}(t)$, so the available energy in the communication region decreases as $R^2(t)$ or $R^{3/2}(t)$. [The available shear energy in a shear-dominated universes would decrease as $R^3(t)$ if horizons were present.] Since the available energy must increase if an infinite amount of information is to be processed and stored, I conclude that horizons will prevent an infinite amount of information processing and storage.

But the disappearance of horizons is only a necessary condition for unlimited communication; it is far from being a sufficient one. Even if horizons disappear, it is still necessary to use energy to transmit the signals in the horizon-free direction. Dyson (1979) has investigated the problem of transmission of signals via electromagnetic radiation in open Friedman universes (his analysis extends immediately to the flat Friedman universe, and to ever-expanding anisotropic universes). Dyson assumed that the transmitter and receiver consisted of N and N' electrons, respectively, and that the comoving size of the receiver was constant [that is, each actual physical length scale of the receiver is increasing as $R(t)$]. He concluded that one could select the bandwidth, transmission frequency, and duty cycle in such a way that an infinite number of bits could be transmitted between two observers of constant comoving separation between now and i^+ with *finite* total expenditure of energy. However, his calculations did indicate that only computer elements that today are within a redshift of about 1.7 of each other (corresponding to a comoving coordinate distance of 1) could send signals directly to each other. Beyond this distance, the energy required for the direct transmission of a bit rises exponentially with comoving radial coordinate. Dyson argued that signals could be sent between observers with arbitrary comoving separation by transmitting through relays, but he did not investigate the energy cost of such relay transmission. It seems likely that the energy cost of each relay would be of the order of kT per bit, so that MkT would be required per bit to transmit a message a comoving distance of M. Thus the energy cost of transmission would rise linearly with the comoving separation. Since the energy available in the form of shear energy in open universes diverges linearly with time, there will be sufficient energy in open universes to transmit an arbitrarily large number of bits an arbitrarily large comoving distance. In flat universes, this is impossible, since only a finite amount of energy is available in a comoving volume.

Dyson's calculations assume that the interstellar medium is completely transparent to radiation, so they do not apply in the regime near the final singularity of a closed universe, where the optical depth is quite small. I shall assume that the transmission of information in this regime is governed by the Shannon formula for the transmission of information in a noisy channel (Shannon and Weaver, 1949): $dI/dt \leq B\ln(1+P_S/P_N)$, where B is the bandwidth of the signal, P_S is the power of the signal, and P_N is the power of the noise. Metzner and Morrison (1959) were the first to apply the Shannon formula in a cosmological context, but they were interested only in the transmission of signals from the past to the present-day Earth.

For small signal-to-noise ratios, the Shannon formula reduces to $dI/dt \leq BP_S/P_N = P_S/kT$, where in the last step I have used the Nyquist relation $P_N = kTB$ (Brillouin, 1962). The expression

$$dI/dt \leqslant P_S/kT = (dE/dT)_S/kT$$

closely resembles equation (30); the only difference is that in (30), the quantities dE/dt and T are measured at the same event, whereas it is not clear how to apply the above expression, since the temperature, signal power, and noise power are all changing between the emission and reception of the signal, due to the cosmological blueshift or redshift. It will most often be a blueshift, because the signals will generally be transmitted in the contracting direction for which the horizon disappears. I shall assume that these cosmological Doppler shifts are properly taken into account by letting dE/dt_S be the power of the transmitter, while T is the temperature of the receiver; this means that the information transmission allowed by the Shannon formula is reduced from that of (30) by the factor $R(t)$, since temperature scales as $T \sim 1/R(t)$. Putting an extra factor of R into the lhs of (37), we see that $I = \int (dI/dt)\, dt$ still diverges, though at the reduced rate $t^{-1/3}$ [it diverges only as $\ln(t)$ if we take into account the waste heat due to information processing and transmission], so it is possible to transmit an infinite number of bits from one side to the other of a closed universe with an omega point.

We now come to condition 3, the requirement that it must be possible to *store* an amount of information that diverges as the c-boundary is approached. The storage of n bits of information requires the existence in space of at least $2n$ distinguishable states of matter, radiation, or black holes. Furthermore, in order that this information not be lost, the energy of these states must be above the random fluctuation energy kT of the environment of the storage device, as I discussed above. It seems unlikely that radiation by itself can serve as a storage device, for it tends to dissipate unless it is confined by solid matter. In all environments I shall be concerned with, the far future of an expanding universe and the hot environment near the final singularity of a closed universe, solid matter will not exist, so radiation is probably ruled out as the basis of information storage. Black holes are probably ruled out as storage devices in the far future for three reasons: first, if Barrow and Tipler (1978) are correct that black holes greater than supercluster mass never form, then black holes will eventually evaporate and hence cease to exist; second, it is not clear how black holes could be used to store distinguishable bits of information; third, if a black hole could be used to store information, the amount of mass-energy used per bit is likely to be too large to be supplied by the feeble energy sources of the far future (recall that the power usage must decrease as $dE/dt \sim t^{-\delta}$). However, I should emphasize that neither radiation nor black holes are conclusively ruled out as information storage devices in the far future, though they do seem unlikely candidates.

This leaves only matter from which to construct information storage devices. If we ignore the exotic forms of matter whose existence has been hypothesized but never seen, the only matter remaining in the far future are positrons and electrons in a mixture of free particle plasma and positronium. Dyson (1981) was the first to suggest that it may be possible for information to be stored in such a medium.

It is certainly possible to store information in a positronium atom. For example, parallel spins of the electron and positron could denote 1, and antiparallel spins could denote 0. The energy ΔE required to induce a transition between the lower energy antiparallel state and the parallel state decreases as $\Delta E \sim 1/n^3$, where n is the principal quantum number of positronium (Bethe and Salpeter, 1957). We must have $\Delta E > kT \sim 1/R(t)$, and from equations (25) and (26) we have $r \sim n^2$, so $r \leq R^{2/3}(t)$. In short, the positronium atoms used to store information must grow, but at a slower rate than the universe expands. The energy needed to cancel out the radiative losses of the positronium and E decreases sufficiently rapidly so that it is possible to satisfy the above constraints on dE/dt in both the flat and open universes and still cause an infinite number of transitions between now and the c-boundary, if we ignore the problem of exactly how the available energy in the form of shear is to be transferred to the atoms. I shall also not deal with the question of whether the atoms can be organized together in the complicated fashion required for computers. These are very complex unsolved problems, which I shall not attack in this preliminary survey. All I can do here is indicate the directions that future research on the question of unlimited computation must take.

As the universe expands, the number of positronium atoms being used as RAM must diverge as the *c*-boundary is approached if the amount of information actually stored is to diverge, and this is a very serious problem, since, as Dyson has shown, the region being used for computation is bounded in comoving coordinates, at least for flat universes. Furthermore, there will be difficulties in open universes in obtaining the necessary positronium, because very little positronium will be formed because of the rapid expansion. Thus I make a very tentative prediction that the universe must be closed if unlimited computation is to be possible.

In open or flat universes, it is necessary that the region in which the information is stored diverge as the *c*-boundary is approached, for Bekenstein (1981a) has shown that the information that can be stored in a region of radius D is bounded above by

$$I \leq 2\pi ED/c\hbar \tag{38}$$

where E is the amount of energy used to store one bit. Bekenstein (1981b) has argued that E is bounded below in any finite region in open and flat universes, so if I is to diverge, so must D. The Bekenstein bound (38) has been derived only for spacetimes with noncompact Cauchy hypersurfaces and for closed universes with event horizons. If it applies to *all* closed universes, then it will be impossible for unlimited information storage to be possible in closed universes, since D will go to zero as the *c*-boundary is approached, and thus (38) would prevent the amount of information stored from diverging. The derivation of (38) seems to depend crucially on the presence of event horizons, so there is no reason to believe it will apply to closed universes with an omega point. I shall assume here that it does not, but this is a point that needs to be investigated. But there are good reasons to believe it will apply (see, however, Page, 1982; Bekenstein, 1983; Unruh and Wald, 1983) to closed universes with horizons. If it does, this provides another argument that unlimited computation requires the *c*-boundary to be an omega point. It also indicates that in order for computation to be unlimited, the computer will ultimately have to encompass the entire universe.

It is occasionally claimed (Mundici, 1981; Bremermann, 1982) that the energy-time uncertainty relation restricts the rate at which computers can process information. Mundici (1981) has claimed that the energy-time uncertainty relations require $(dI/dt)^2 \leq \hbar^{-1} dE/dt$. Bremermann (1982), on the other hand, feels that the energy-time uncertainty relations require

$dI/dt \leq [h^{-1}\ln(1+4\pi)]E$. However, it is not clear that such a restriction actually applies, because it is not clear what time coordinate t is the appropriate one to use (Jammer, 1974). In fact, if t is a time external to the system being measured, then the energy-time uncertainty relation $\Delta E\,\Delta t \geq \hbar/2$ can be evaded (Caves et al., 1980; Likharev, 1982). The energy-time relation only restricts the measurement of times that are intrinsically defined by the physical system being measured. Landauer (1982) has given other arguments which suggest that the energy-time uncertainty relation in fact does not restrict the information processing rate.

Nevertheless, let us assume that the information processing rate is in fact restricted by either the Mundici relation or the Bremermann relation. I shall now show that Mundici's relation will not prevent an infinite amount of information processing in any type of universe, be it open, flat, or closed. Bremmermann's relation will prevent infinite information processing only in the flat universe. If we assume that the t which is restricted is cosmic proper time, then a straightforward calculation shows that $I=\int (dI/dt)\,dt$ can diverge even if $(dI/dt)^2 \leq \hbar^{-1}\,dE/dt$ (Mundici) or $dI/dt \leq h^{-1}E$ (Bremermann) applies in open universes (since $dE/dt \sim$ const, or $E \sim t$), or in closed universes (since $dE/dt \sim t^{-2}$, or $E \sim t^{-1}$). In flat universes Mundici's relation will hold with infinite information processing if $1 < \delta \leq 2$ (since $dE/dt \sim t^{-\delta}$). Bremermann's relation prevents an infinite amount of information processing in the flat universe case because his relation requires an infinite amount of energy for an infinite amount of information processing, and there is only a finite amount of energy available in a finite comoving volume in the flat universe case.

In closed universes condition 3 requires that information be stored in high-energy particle states of mass m. As the radius of a closed universe near the final state goes to zero near the final singularity, the information must be stored in particle states of higher and higher energies in order that it not be lost through random fluctuations. Furthermore, the total number N of particle states of mass m in the closed universe must diverge as the final singularity is approached if the amount of information stored in these particle states is to diverge, but the divergence of the total energy in elementary particle states cannot be more rapid than the divergence of the shear energy which is the energy source for the creation of these particle states. These are clearly necessary conditions for unlimited information to be stored, though they are not sufficient conditions. However, these conditions suffice to derive some restrictions on elementary particle states.

The restriction that the mass of the elementary particle state be greater than the thermal energy is expressed as

$$m > kT \sim 1/R(t) \tag{39}$$

while the requirement that the energy in the particle states be less than the shear energy can be written in the form of a restriction on energy densities:

$$Nm/V < \sigma^2 \sim 1/t^2 \sim 1/R^6 \tag{40}$$

where I have used the growth rate of shear energy density in the last two steps. Now $V \sim R^3$, so (40) becomes

$$Nm/V \geq N(1/R)/R^3 \sim N/t^{4/3} \tag{41}$$

so the total number of particle states could grow as fast as $1/t^{2/3}$ without violating the energy upper bound. The total stored information I_{TOT} we would expect to grow roughly as N, so I_{TOT} can diverge as fast as $1/t^{2/3}$ if the growth of particle states with energy permits. But the energy in the particle states cannot grow faster than this without exhausing the energy supply. Suppose that we write $N \sim t^{-\varepsilon}$, where $0 < \varepsilon < 2/3$. Remembering that almost all of the time $R(t) \sim t^{1/3}$ near the final singularity, we obtain from (41)

$$m < V/Nt^2 \sim 1/Nt \sim t^{\varepsilon - 1} \tag{42}$$

The inequalities (39) and (42) can be combined to give a constraint on the mass-energy of the particles:

$$1/t^{1/3} < m < t^{\varepsilon - 1} \tag{43}$$

We can put the energy scales into (43) by noting that $1/t^{1/3} \sim 1/R(t) \sim kT \sim E$, where E is the actual particle energy measured in GeV. The inequality (43) then becomes

$$E < m < E^{3(1-\varepsilon)} \tag{44}$$

where $0 < \varepsilon < 2/3$. The final inequality (44) means that if condition 3 is to be satisfied, there must be a particle state with energy in between the upper and lower limits of (44). Furthermore, on the average the number of particle states cannot grow faster than E^3, since otherwise the shear would be damped out by the production of particle states.

The bound (44) is not incredibly strong, but it is sufficient to rule out a number of proposed elementary particle spectra at high energies, for example, the exponentially increasing spectra that underlie the Hagedorn equation of state (Weinberg, 1972). It rules out the possibility of a "great desert"—a lack of particle states—between the electroweak unification energy and the grand unification energy. It would also rule out standard superstring theories, for in such theories the energy going into the excitation of the vibration modes of the string grow too fast above the Planck energy.

The true importance of (44) is that it shows it is possible to *test* the possibility of unlimited computation, for computation in the far future is possible only if matter has certain properties, and if we assume that the properties of matter do not change with time, then these properties of matter in the far future will also be properties of matter now. It is not possible of course to investigate the properties of matter and the structure of the universe in the far future, but it is possible to investigate these aspects of nature today.

I showed in Section 2 that if computers are to be able to continue to communicate indefinitely near the final singularity, the *c*-boundary must be a single point, the omega point. Furthermore, solutions of the Einstein equations with an omega point are probably of measure zero in the initial data of the space of solutions. That is, it is of measure zero if the action of computing machines on the universe is neglected. In principle, may be possible by exerting relatively small forces at just the right series of instants on a truly global scale for machines to *force* a generic Bianchi type IX closed universe into having an omega point by systematically eliminating the horizons in sequence in all directions an infinite number of times. This will be possible, that is, if the operations of these machines encompass the entire universe, and if the properties of matter will allow the necessary forces. Since the probability is one that the actual universe is not of measure zero in the space of solutions, then if unlimited computation is possible and the universe is closed, it must be true that matter has the appropriate properties. Determining these properties would yield another testable prediction.

I have ignored quantum gravity effects in the above discussion, for there is at present no generally accepted theory of quantum gravity. But of course such effects will be of great importance near the final singularity of a closed universe. However, a few general remarks can be made. If information processing and storage go on without limit, then even in quantum cosmoslogy the physical time *must* be unidirectional. In particular, closed universes—classical or quantum—in which the entropy decreases during the contracting phase of the universe are ruled out. This means that the first version of Hawking's proposed quantum cosmology (1984) is ruled out, because entropy does in fact decrease in its contracting phase. (In a later version, the entropy increases monotonically.) Similarly, any classical or quantum cosmology based on a compact *four*-dimensional compact topology is ruled out because this would imply a cyclic time.

I pointed out in Section 2 that only in a closed universe is it possible for *all* timelike curves to be forever in causal contact with one another. For this reason, the universe must be closed if computation is to be truly unlimited, but this is only a weak prediction. Nevertheless, let me assume that the universe is closed, and follow broad features of the evolution of a universal Turing machine from a time near the present all the say into the Omega Point. (It is possibe for a closed universe to have an omega point, but for computation to cease before it is reached. I shall distinguish between these two classes of universes with omega points by capitalizing "Omega Point" if computation continues all the way into the omega point.)

The computer begins its expansion from a single planet. The information stored and the material available for computer elements increases as t^3 initially, but eventually the rate of increase drops to t^2 as the resources in the center of the expanding computation region are exhausted. The increase will continue until the computer has expanded to encompass fully one-half of the entire universe. Because of the curvature of space (assumed to be a three-sphere), the proper volume of the computer will *decrease* as it expands. But if the universe is sufficiently large, the region in which the computer is growing can still increase because the energy source is no longer matter energy, but shear energy. It is likely that before the computer has been able to expand to cover more than half the universe, the contraction will have begun. The cosmological radiation temperature, which has dropped to extremely low values, will begin to increase again. The redshift will have become a blueshift. The information will continue to increase as t^n in the early contracting phase, where t is as before the proper time from the initial singularity, and n is some number ≤ 3. The value of n will depend on which specific energy source is used, what form of matter is used to store information, and what percentage of the entire universe is encompassed by the computer.

Finally, the time is reached when the computer has encompassed the entire universe and gained control of all matter contained therein. The computer begins to manipulate the dynamical evolution of the universe as a whole, forcing the horizons to disappear, first in one direction, and then another. The information stored continues to increase, but now at the average rate $t^{-\varepsilon}$, where t is the proper time until the final singularity at $t = 0$, and $0 < \varepsilon < 2/3$. As measured in proper time, the rate of growth of stored information is faster than exponential growth (since it diverges in finite time), but a more accurate measure of subjective time in this epoch is the amount of time needed to process one bit. In this time measure, the information storage increases, but the increase is a power law, with the power less than ε. However, the increase continues as a power law indefinitely.

In summary, it seems that a true universal Turing machine could in principle be constructed in the actual universe, or equivalently, that computation and information storage is not limited in principle, provided that (1) the universe is closed, (2) the final singularity is an omega point, and (3) the particle spectrum has a certain form. It will be interesting to discover if the universe has these properties.

ACKNOWLEDGMENTS

I am grateful to D. Deutsch, D. N. Page, and R. A. Matzner for helfpful discussions. I am especially grateful to John D. Barrow, my coauthor of *The Anthropic Cosmological Principle.* This research was supported in part by the National Science Foundation under grant Phys-84-09672.

REFERENCES

Barrow, J. D. and Matzner, R. A. (1977). *Monthly Notices of the Royal Astronomical Society,* **180,** 317.

Barrow, J. D., and Tipler, F. J. (1978). *Nature,* **276,** 453.

Barrow, J. D., and Tipler, F. J. (1985). *Monthly Notices of the Royal Astronomical Society,* **216,** 395.

Barrow, J. D., and Tipler, F. J. (1986). *The Anthropic Cosmological Principle,* Oxford University Press, Oxford.

Barrow, J. D., and Turner, M. S. (1970). *Nature,* **292,** 35.

Bekenstein, J. D. (1981). *Physical Review Letters,* **46,** 623.

Bekenstein, J. D. (1981a). *Physical Review D,* **23,** 287.

Bekenstein, J. D. (1983). *Physical Review D,* **27,** 2262.

Bennett, C. H. (1973). *IBM Journal of Research and Development,* **17,** 525.

Bennett, C. H. (1982). *International Journal of Theoretical Physics,* **21,** 905.

Bennett, C. H., and Landauer, R. (1985). *Scientific American,* **253**(July), 48.

Bethe, H. A., and Salpeter, E. E. (1957). *Quantum Mechanics of One- and Two-Electron Atoms,* Academic Press, New York, p. 115.

Bremermann, (1982). *International Journal of Theoretical Physics,* **21,** 203.

Brill, D. R., and Glaherty, F. (1976). *Communications in Mathematical Physics,* **50,** 157.

Brillouin, L. (1962). *Science and Information Theory,* Academic Press, New York.

Bucic, R., and Sachs, R. K. (1976). *General Relativity and Gravitation,* **7,** 21.

Caves, C. M., Thorne, K. S., Drever, R P., Sandberg, V. D., and Zimmermann, M. (1980). *Reviews of Modern Physics,* **52,** 341.

Chandrasekhar, S. (1960). *Principles of Stellar Dynamics,* Dover, New York.

Collins, C. B., and Hawking, S. W. (1973). *Astrophysical Journal,* **180,** 317.

Dicus, D. A., Letaw, J. R., Teplitz, D. C., and Teplitz, V. L. (1982). *Astrophysical Journal,* **252,** 1.

Dicus, D. A., Letaw, L. R., Teplitz, D. C., and Teplitz, V. L. (1983). *Scientific American,* **248**(3), 90.

Dyson, F. J. (1979). *Reviews of Modern Physics,* **51,** 447.

Dyson, F. J. (1981). Life in the Universe, unpublished Darwin Lecture, Darwin College, Cambridge, England, November 10, 1981.

Ellis, G. R. R. (1973). In *Cargese Lectures in Physics,* Volume 6, E. Schatzman, ed., Gordon and Breach, New York.

Feinberg, G. (1981). *Physical Review, D,* **23,** 3075.

Frautschi, S. (1982). *Science,* **217,** 593.

Gerhardt, C. (1983). *Communications in Mathemaitcal Physics*, **89**, 523.
Geroch, R. P. (1970). *Journal of Mathematical Physics*, **11**, 437
Gibbons, G. W., Hawking, S. W., Horowitz, G. T., and Perry, M. J. (1983). *Communications in Mathemaitcal Physics*, **88**, 295.
Hawking, S. W. (1974). *Nature*, **248**, 30.
Hawking, S. W. (1975). *Communications in Mathematical Physics*, **43**, 199.
Hawking, S. W. (1984). *Nuclear Physics B*, **239**, 257.
Hawking, S.W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.
Horowitz, G. T., and Perry, M. J. (1982). *Physical Review Letters*, **48**, 371.
Islam, J. N. (1977). *Quarterly Journal of the Royal Astronomical Society*, **18**, 3.
Islam, J. N. (1979). *Sky & Telescope*, **57**, 13.
Jammer, M. (1974). *The Philosophy of Quantum Mechanics*, Wiley, New York.
Landau, L., and Lifshitz, E. M. (1975). *The Classical Theory of Fields*, 4th ed., Pergamon, New York.
Landauer, R. (1961). *IBM Journal of Research and Development*, **3**, 183
Landauer, R. and Woo, J. W. F. (1971). *Journal of Applied Physics*, **42**, 2301.
Landauer, R. (1982). *International Journal of Theoretical Physics*, **21**, 283.
Landauer, R. (1985). *Annals of the New York Academy of Sciences*, **426**, 161.
Likharev, K. K. (1982). *International Journal of Theoretical Physics*, **21**, 311.
Marsden, J. E., and Tipler, F. J. (1980). *Phys Reports*, **66**, 109
Matzner R. A. (1971). *Annals of Physics*, **65**, 482.
Metzner, A. W. K. and Morrison, P. (1959). *Monthly Notices of the Royal Society*, **119**, 657.
Misner, C. W. (1968). *Astrophysical Journal*, **151**, 431
Misner, C. W. (1969). *Physical Review*, **186**, 1328.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
Mundici, D. (1981). *Nuovo Cimento*, **61B**, 297.
Page, D. N. (1982). *Physical Review D*, **26**, 947.
Page, D. N., and McKee M. R. (1981). *Physical Review D*, **24** 1458.
Page, D. N., and McKee, M. R. (1983). *Mercury*, **12**, 17.
Porod, W., Grondin, R. O., Ferry, D. K., and Porod, G. (1984a). *Physical Review Letters*, **52**, 232.
Porod, W., Grondin, R. O., Ferry, D. K., and Porod, G. (1984b). *Physical Review Letters*, **53**, 1202
Rees, M. (1969). *Observatory*, **89**, 193.
Ryan, M. P., and Shepley, L. C. (1975). *Homogeneous Relativistic Cosmologies*, Princeton University Press, Princeton, New Jersey.
Sakurai, J. J. (1967). *Advanced Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts, p. 44.
Saslaw, W. C. (1973). *Publications Astronomical Society of the Pacific*, **85**, 5.
Saslaw, W. C (1974). In *IAU Symposium No.* 58, J. R. Shakesaft, ed., Reidel, Dordrecht.
Schoen, R., and Yau, S.-T. (1978). *Communications in Mathematical Physics*, **65**, 45.
Schoen, R. and Yau, S.-T. (1979). *Physical Review Letters*, **42**, 547.
Seifert, H.-J. (1971). *General Relativity and Gravitation*, **1**.
Shannon, C. E., and Weaver, W. (1949). *The Mathematical Theory of Communication*, University of Illinois Press, Urbana, Illinois.
Spitzer, L., and Hära, R. (1958). *Astrophysical Journal*, **127**, 544.
Thorne, K. S. (1967). *Astrophysical Journal*, **148**, 51.
Tipler, F. J. (1977). *Physical Review D*, **15**, 942.
Tipler, F. J. (1979a). *Nature*, **280**, 203.

Tipler, F. J. (1979b). *General Relativity and Gravitation*, **10**, 1005.
Tipler, F. J. (1980). In *Essays in General Relativity*, F. J. Tipler, ed., Academic Press, New York.
Tipler, F. J. (1986). *Journal of Mathematical Physics*, **27**, 559.
Toffoli, T., Bennett, C. H., Landauer, R., Feynman R. P., *et al.* (1982). *Proceedings of the Conference on Physics and Computation, International Journal of Theoretical Physics*, **21**(3/4, 6/7, 12).
Trimble, V. (1975). *Review of Modern Physics*, **47**, 877.
Unruh, W. G. and Wald, R. M. (1983). *Physical Review D*, **27**, 2271.
Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.
Witten, E. (1981). *Communications in Mathematical Physics*, **80**, 381.
Zel'dovich, Ya. B., and Novikov, I. D. (1971). *Relativistic Astrophysics*, Volume I: *Stars and Relativity*, University of Chicago Press, Chicago, pp. 337–338.
Zurek, W. H. (1984). *Physical Review Letters*, **53**, 391.